Notes on anti s-fuzzy subfields of a field
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ABSTRACT
In this paper, we made an attempt to study the algebraic nature of anti S-fuzzy subfield of a field.

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Introduction
After the introduction of fuzzy sets by L.A.Zadeh[15], several researchers explored on the generalization of the concept of fuzzy sets. The notion of fuzzy subgroups, anti-fuzzy subgroups, fuzzy fields and fuzzy linear spaces was introduced by Biswas.R[4, 5]. In this paper, we introduce the some

Definition:
Let X be a non-empty set. A fuzzy subset A of X is a function A : X → [0, 1].

Definition:
A S-norm is a binary operation S : [0, 1]×[0, 1] → [0, 1] satisfying the following requirements:
(i) S(0, x) = x, S(1, x) = 1 (boundary condition)
(ii) S(x, y) = S(y, x) (commutativity)
(iii) S(x, y) ≥ S (S(x, y), z ) (associativity)
(iv) if x ≤ y and w ≤ z, then S(x, w ) ≤ S (y, z ) (monotonicity).

Definition:
Let ( F, +, · ) be a field. A fuzzy subset A of F is said to be an anti S-fuzzy subfield ( anti fuzzy subfield with respect to S-norm ) of F if the following conditions are satisfied:
(i) A(x+y) ≤ S (A(x), A(y) ), for all x and y in F,
(ii) A(− x) ≤ A(x), for all x in F,
(iii) A(xy ) ≤ S (A(x), A(y) ), for all x and y in F,
(iv) A(x−1) ≤ A(x), for all x ≠ 0 in F, where 0 is the additive identity of F.

Definition:
Let ( F, +, · ) and ( F', +, · ) be any two fields. Let f : F → F' be any function and A be an anti S-fuzzy subfield of F, V be an anti S-fuzzy subfield in f(F) = F', defined by V(y) = inf_{x ∈ f−¹(y)} A(x), for all x in F and y in F'. Then A is called a preimage of V under f and is denoted by f−¹(V).

Definition:
Let A and B be any two fuzzy subsets of sets G and H, respectively. The anti-product of A and B, denoted as AxB, is defined as AxB = { ( ( x, y ), AxB( x, y ) ) | for all x in G and y in H }, where AxB( x, y ) = max { A(x), B(y) }, for all x in G and y in H.

Definition:
Let A be a fuzzy subset in a set S, the strongest fuzzy relation on S, that is a fuzzy relation on A is V= {((x,y), V(x,y)) / x and y in S } given by V(x, y) = max { A(x), A(y) }, for all x and y in S.

Definition:
Let A be an anti S-fuzzy subfield of a field ( F, +, · ) and a in F. Then the pseudo anti S-fuzzy coset (aA)^D is defined by ((aA)^D)(x) = p(a)A(x), for every x in F and for some p in P.

Properties:
Theorem: If A is an anti S-fuzzy subfield of a field ( F, +, · ), then A(− x) = A(x), for all x in F and A(x−1) = A(x), for all x ≠ 0 in F and A(x) ≥ A(0), for all x in F and A(x) ≥ A(1), for all x in F, where 0 and 1 are identity elements in F.

Proof:
For x in F and 0, 1 are identity elements in F. Now, A(x) = A(− (− x)) = A(− x) = A(x). Therefore, A(x) = A(x), for all x in F. And, A(x) = A(x−1) = A(−1)x ≤ A(x). Therefore, A(x−1) = A(x), for all x ≠ 0 in F. And, A(0) = A(−x−1) ≤ S (A(x), A(−x−1) ) = A(x). Therefore, A(0) ≤ A(x), for all x in F. And, A(1) = A(xx−1) ≤ S(A(x), A(x−1)) = A(x). Therefore, A(1) ≤ A(x), for all x ≠ 0 in F.

Theorem: If A is an anti S-fuzzy subfield of a field ( F, +, · ), then
(i) A(x−y) = A(0) gives A(x) = A(y), for all x and y in F,
(ii) A(xy) = A(1) gives A(x) = A(y), for all x and y ≠ 0 in F, where 0 is the additive identity of F.

Proof:
Let x and y in F and 0, 1 are identity elements in F. (i) Now, A(x) = A(x−y+y) ≤ S (A(x−y), A(y) ) = S (A(0), A(y) ) = A(y) = (A(x−(x−y))) ≤ S(A(x−y), A(x−y) ) = S (A(0), A(x) ) = A(x), Therefore, A(x) = A(y), for all x and y in F. (ii) Now, A(x) = A(xy) ≤ S(A(xy), A(y) ) = S (A(1), A(y) ) = A(y) = A(x−(x−y)) ≤ S (A(x−y), A(x−y) ) ≤ S (A(0), A(x) ) = A(x), Therefore, A(x) = A(y), for all x and y ≠ 0 in F.

Theorem: Let A be a fuzzy subset of a field ( F, +, · ). If A(e) = A(e') = 0, A(x−y) ≤ S (A(x), A(y) ), for all x and y in F and A(xy)^D ≤ S (A(x), A(y) ), for all x and y ≠ e in F, then A is an anti S-fuzzy subfield of F, where e and e' are identity elements of F.

Proof:
Let e and e' be identity elements of F and x and y in F. Now A(x−y) = A(e−x) ≤ S (A(e), A(x) ) = S (A(0), A(x) ) = A(x), Therefore, A(x−y) ≤ S (A(e), A(x) ) = S (O, A(x) ) = A(x), Therefore, A(xy)^D ≤ S (A(x), A(y) ), for all x and y on F. And A(ax+y) = A(x−(x−y)) ≤ S(A(x), A(y) ) ≤ S (A(x), A(y) ) = A(x), Therefore, A(xy) ≤ S (A(x), A(y) ), for all x and y in F. And A(x−y)^D ≤ S (A(x), A(y) ) ≤ S (A(x), A(y) ) ≤ S (A(x), A(y) ) ≤ S (A(x), A(y) ). Therefore, A(xy)^D ≤ S (A(x), A(y) ), for all x and y in F.
(A(x), A(y)). Therefore, A(xy) ≤ S(A(x), A(y)), for all x and y ≠ e in F. Hence A is an anti S-fuzzy subfield of F.

**Theorem:** If A is an anti S-fuzzy subfield of a field \( F, (+, \cdot) \), then \( H = \{ x / x_\in F : A(x) = 0 \} \) is either empty or is a subfield of F.

**Proof:** If no element satisfies this condition, then H is empty. If x and y in H, then \( A(x) = A(y) = 0 \). Therefore, A\((x+y)\) = 0, for all x and y in F. Hence, \( H = \{ x / x_\in F : A(x) = 0 \} \) is either empty or is a subfield of F.

**Theorem:** If A is an anti S-fuzzy subfield of a field \( F, (+, \cdot) \), then \( H = \{ x \in F : A(x) = A(e) \} \) is either empty or is a subfield of F, where e and e′ are identity elements of F.

**Proof:** If no element satisfies this condition, then H is empty. If x and y satisfies this condition, then A\((x+y)\) = A(e), for all x and y in F. Hence, A\((x+y)\) = A(e), for all x and y in F. Therefore, A\((x+y)\) = S(A(x), A(y)).

**Theorem:** Let A be an anti S-fuzzy subfield of a field \( F, (+, \cdot) \). Then \( A(x) = A(y) = 0 \), for all x and y ≠ e in F. Hence A is either empty or is a subfield of F.

**Proof:** Let x and y in F. By the definition A\((x+y)\) = S(A(x), A(y)), which implies that 1 ≤ S(A(x), A(y)). Therefore, either A\((x+y)\) = 1 or A\((x+y)\) = 0, for all x and y in F. By the definition A\((x+y)\) = S(A(x), A(y)), which implies that 1 ≤ S(A(x), A(y)). Therefore, either A\((x+y)\) = 1 or A\((x+y)\) = 0, for all x and y ≠ e in F.

**Theorem:** Let (F, +, \cdot) be a field. If A is an anti S-fuzzy subfield of F, then A\((x+y)\) = S(A(x), A(y)), for all x and y in F and A\((xy)\) = S(A(x), A(y)), for all x ≠ 0 and y ≠ 0 in F. Hence A is an anti S-fuzzy subfield of GxH. **Proof:** Let A be an anti S-fuzzy subfield of a field \( F, (+, \cdot) \). Then A\((x+y)\) = S(A(x), A(y)), for all x and y in F and A\((xy)\) = S(A(x), A(y)), for all x ≠ 0 and y ≠ 0 in F. Hence A is an anti S-fuzzy subfield of GxH.
\( B(y_1), B(y_2) ) = S \left( \max (A(x_1), B(y_1)) , \max (A(x_2), B(y_2)) \right) = S(AxB(x_1, y_1), AxB(x_2, y_2)). \)

Therefore, \( AxB(x_1, y_1) - (x_2, y_2) \leq \max (AxB(x_1, y_1), AxB(x_2, y_2)) \), for all \( x_1 \) and \( x_2 \) in \( G \) and \( y_1 \) and \( y_2 \) in \( H \). And, \( AxB(x_1, y_1), (x_2, y_2)^2 \} = \max (AxB(x_1, y_1), B(y_2)) \), \( \max (S(A(x_1), B(y_1)), S(A(x_2), B(y_2))) = S(\max (A(x_1), B(y_1)), \max (A(x_2), B(y_2))) = \max (S(A(x_1), B(y_1)), S(A(x_2), B(y_2))). \)

Therefore, \( AxB(x_1, y_1), (x_2, y_2)^2 \} = \max (S(A(x_1), B(y_1)), S(A(x_2), B(y_2))), for all \( x_1 \) and \( x_2 \) not \( 0 \) in \( G \) and \( y_1 \) and \( y_2 \) not \( 0 \) in \( H \). Hence anti-product \( AxB \) is an anti \( S \)-fuzzy subfield of \( GxH \).

**Theorem:** Let \( A \) and \( B \) be fuzzy subsets of the fields \( G \) and \( H \), respectively. Suppose that \( 0 \), \( 1 \), \( 0, 1 \), are the identity elements of 

(i) \( B(0) \leq A(x) \), for all \( x \) in \( G \) and \( B(1) \leq A(x) \), for all \( x \neq 0 \) in \( G \).

(ii) \( A(0) \leq B(y) \), for all \( y \) in \( H \) and \( A(1) \leq B(y) \), for all \( y \neq 0 \) in \( H \).

**Proof:** Let the anti-product \( AxB \) be an anti \( S \)-fuzzy subfield of \( GxH \). By contraposition, suppose that none of the statements (i) and (ii) holds. Then we can find \( a \) in \( G \) and \( b \) in \( H \) such that \( A(a) < B(0) \), \( A(a) < B(1) \) and \( B(b) < A(0), B(b) < A(1) \). We have, \( AxB(a, b) = \max (A(a), B(b)) < \max (A(0), B(0)) = \max (0, 0) = \max (A(1), B(1)) = \max (A(a), B(b)). \)

Thus anti-product \( AxB \) is not an anti \( S \)-fuzzy subfield of \( GxH \). Hence either \( B(0) \leq A(x) \), for all \( x \) in \( G \) and \( B(1) \leq A(x) \), for all \( x \neq 0 \) in \( G \).

**Theorem:** Let \( A \) and \( B \) be fuzzy subsets of the fields \( G \) and \( H \), respectively and the anti-product \( AxB \) is an anti \( S \)-fuzzy subfield of \( GxH \). Then the following are true:

(i) \( A(x, 1) \geq A(x) \), \( A(x) \geq A(1) \), and \( A(1) \geq A(x) \) for all \( x \) in \( G \) and \( y \) in \( H \).

(ii) \( A(x, y) = S(\max (A(x), A(y)), \max (A(x), A(y))) = S(\max (S(A(x), A(y)), S(A(x), A(y)))) \), \( \max (A(x, y), A(x, y)) = S(\max (S(A(x), A(y)), S(A(x), A(y)))) \), \( \max (A(x, y), A(x, y)) = S(\max (S(A(x), A(y)), S(A(x), A(y)))) \), \( A(x, y) = S(\max (A(x), A(y)), \max (A(x), A(y))) = S(\max (S(A(x), A(y)), S(A(x), A(y)))) \).

Hence \( B \) is an anti \( S \)-fuzzy subfield of \( H \). Thus (ii) is proved. And (iii) is clear.

**Theorem:** Let \( A \) be a fuzzy subset of a field \( F, +, \cdot \) and \( B \) be the anti-strongest \( S \)-fuzzy relation of \( F \). Then \( A \) is an anti \( S \)-fuzzy subfield of \( F \) if and only if \( V \) is an anti \( S \)-fuzzy subfield of \( FxF \).

**Proof:** Suppose that \( A \) is an anti \( S \)-fuzzy subfield of \( F \). Then for any \( x = (x_1, x_2), y = (y_1, y_2) \) in \( F \). We have, \( V(x+y) = V((x_1, x_2) − (y_1, y_2)) = V(x_1 − y_1, x_2 − y_2) \geq \max (A(x_1), y_1), A(x_2) \leq \max (A(x_1), A(x_2)) \).

Therefore, \( V(x+y) = S(V(x, x) - (y_1, y_2)) = S(V(x, x) - (y_1, y_2)) , \) for all \( x \) and \( y \) in \( F \). And \( V((x_1, x_2) − (y_1, y_2)) = V((x_1, x_2) − (y_1, y_2)) , \) for all \( x \) and \( y \) in \( F \).

Therefore, \( V(x+y) = S(V(x, x) - (y_1, y_2)) = S(V(x, x) - (y_1, y_2)) , \) for all \( x \) and \( y \) in \( F \). And \( V((x_1, x_2) − (y_1, y_2)) = V((x_1, x_2) − (y_1, y_2)) , \) for all \( x \) and \( y \) in \( F \).

Therefore, \( V(x+y) = S(V(x, x) - (y_1, y_2)) = S(V(x, x) - (y_1, y_2)) , \) for all \( x \) and \( y \) in \( F \). And \( V((x_1, x_2) − (y_1, y_2)) = V((x_1, x_2) − (y_1, y_2)) , \) for all \( x \) and \( y \) in \( F \).

Hence \( A \) is an anti \( S \)-fuzzy subfield of \( F \).}

**Theorem:** Let \( A \) be an anti \( S \)-fuzzy subfield of a field \( F, +, \cdot \) and \( f \) be an isomorphism from a field \( F \) onto \( H \). Then \( A \circ f \) is an anti \( S \)-fuzzy subfield of \( F \times F \).

**Proof:** Suppose that \( A \) is an anti \( S \)-fuzzy subfield of \( F \). Then for any \( x = (x_1, x_2), y = (y_1, y_2) \) in \( F \). We have, \( V(x+y) = V((x_1, x_2) − (y_1, y_2)) = V((x_1, x_2) − (y_1, y_2)) , \) for all \( x \) and \( y \) in \( F \). And \( V((x_1, x_2) − (y_1, y_2)) = V((x_1, x_2) − (y_1, y_2)) , \) for all \( x \) and \( y \) in \( F \).

Therefore, \( V(x+y) = S(V(x, x) - (y_1, y_2)) = S(V(x, x) - (y_1, y_2)) , \) for all \( x \) and \( y \) in \( F \).

Hence \( V(x+y) = S(V(x, x) - (y_1, y_2)) = S(V(x, x) - (y_1, y_2)) , \) for all \( x \) and \( y \) in \( F \).

Therefore, \( V(x+y) = S(V(x, x) - (y_1, y_2)) = S(V(x, x) - (y_1, y_2)) , \) for all \( x \) and \( y \) in \( F \).

Therefore, \( V(x+y) = S(V(x, x) - (y_1, y_2)) = S(V(x, x) - (y_1, y_2)) , \) for all \( x \) and \( y \) in \( F \).

Therefore, \( V(x+y) = S(V(x, x) - (y_1, y_2)) = S(V(x, x) - (y_1, y_2)) , \) for all \( x \) and \( y \) in \( F \).

Therefore, \( V(x+y) = S(V(x, x) - (y_1, y_2)) = S(V(x, x) - (y_1, y_2)) , \) for all \( x \) and \( y \) in \( F \).

Therefore, \( V(x+y) = S(V(x, x) - (y_1, y_2)) = S(V(x, x) - (y_1, y_2)) , \) for all \( x \) and \( y \) in \( F \).

Therefore, \( V(x+y) = S(V(x, x) - (y_1, y_2)) = S(V(x, x) - (y_1, y_2)) , \) for all \( x \) and \( y \) in \( F \).
**Proof:** Let \( x \) and \( y \) in \( F \) and \( A \) be an anti S-fuzzy subfield of a field \( H \). Then we have, \((A\circ f)(x - y) = A(f(x) + f(-y)) = A(f(x) - f(y)) \leq S(A(f(x)), A(f(y))) \leq S((A\circ f)(x), (A\circ f)(y))\), which implies that \((A\circ f)(x - y) \leq S((A\circ f)(x), (A\circ f)(y))\), for all \( x \) and \( y \) in \( F \). And, \((A\circ f)(xy^{-1}) = A(f(xy^{-1})) = A(f(x)f(y^{-1})) = S(A(f(x)), A(f(y))) \leq S((A\circ f)(x), (A\circ f)(y))\), which implies that \((A\circ f)(xy^{-1}) \leq S((A\circ f)(x), (A\circ f)(y))\), for all \( x \) and \( y \neq 0 \) in \( F \). Therefore \((A\circ f)\) is an anti S-fuzzy subfield of a field \( F \).

**Theorem:** If \( A \) is an anti S-fuzzy subfield of a field \((F, +, \cdot)\), then the pseudo anti S-fuzzy coset \((aA)^p\) is an anti S-fuzzy subfield of a field \( F \), for every \( a \in F \).

**Proof:** Let \( A \) be an anti S-fuzzy subfield of a field \((F, +, \cdot)\). For every \( x \) and \( y \) in \( F \), we have, \(((aA)^p)(x - y) = p(a)A(x - y) \leq p(a)S(A(x), A(y)) = S((aA)^p)(x), (aA)^p(y))\). Therefore, \(((aA)^p)(x - y) \leq S((aA)^p)(x), (aA)^p(y))\), for all \( x \) and \( y \) in \( F \). And for every \( x \) and \( y \neq 0 \) in \( F \), \(((aA)^p)(xy^{-1}) = p(a)A(xy^{-1}) \leq p(a)S(A(x), A(y)) = S((aA)^p)(x), (aA)^p(y))\). Therefore, \(((aA)^p)(xy^{-1}) \leq S((aA)^p)(x), (aA)^p(y))\), for all \( x \) and \( y \neq 0 \) in \( F \). Hence \((aA)^p\) is an anti S-fuzzy subfield of a field \( F \).

**Reference**