A common fixed point theorem for weakly compatible mappings satisfying a new contractive condition of integral type in 2-metric space

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ABSTRACT
In this paper we prove a unique common fixed point theorem in 2-metric space. The existence of fixed point for two weakly compatible maps into 2-metric space is established under new contractive condition of integral type by using another functions ϕ and ψ.

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Introduction
As a generalization of the area function for Euclidean triangles the concept of 2-metric introduced by Menger [1] was investigated by Gahler in series of papers [2],[3],[4].

Fixed point theorems in 2-metric spaces have been established by several authors (see e.g.[5],[6],[7],[8],[9]).
In 1986 Jungck [10] introduced the concept of compatible mappings and used to obtain results which generalize a theorem by Park and Bae [11], a theorem by Hadzic [12], and others. Fixed point theorems for compatible mappings and weakly compatible mappings have been established by several authors (see e.g.[13],[14],[15],[16]).

In 2002 Branciari [17] obtained a fixed point result for a single mapping satisfying an analogue of a Banach contraction principle for integral type in the following theorem.

In (2011) V.Gupta and Vareen Manic[18] study the existence and uniqueness of common fixed point theorem for two weakly compatible maps under contractive condition of integral type.

Theorem 1. (Branciari). Let \((X,d)\) be a complete metric space, \(c \in [0,1)\) and let \(T : X \rightarrow X\) be a mapping such that for each \(x,y \in X\),

\[
\int_0^d(x,y) \phi(t) dt \leq c \int_0^d(x,y) \phi(t) dt
\]

where \(\phi : [0,\infty) \rightarrow [0,\infty)\) is a Lebesgue- integrable mapping which is summable (i.e. with finite integral) on each compact subset of \([0,\infty)\), and such that for each \(\epsilon > 0\),

\[
\int_0^\epsilon \phi(t) dt > 0
\]

then \(T\) has a unique fixed point \(a \in X\) such that for each \(x \in X\),

\[
\lim_{n \to \infty} T^n x = a
\]

After the paper of V.Gupta, a lot of research works have been carried out on generalizing contractive condition of integral type for different contractive mappings satisfying various known properties, in metric space and 2-metric space.

The aim of this paper is to translate the works of V.Gupta and Manic[18] from metric space into 2-metric space.

Definitions and Preliminaries

Definition 2. Let \(X\) be a non empty set. A real valued function \(d : X \times X \times X\) is said to be 2-metric in \(X\) if

(i) Each pair of distinct points \(x,y \in X\), there exists a point \(z \in X\) such that \(d(x,y,z) \neq 0\).

(ii) \(d(x,y,z) = 0\) if and only if \(x = y = z\).

(iii) \(d(x,y,z) = d(y,z,x) = d(x,z,y)\)

(iv) \(d(x,y,z) \leq d(x,y,w) + d(x,w,z) + d(w,y,z)\) for all \(x,y,z,w \in X\). When \(d\) is 2-metric on \(X\), then the pair \((X,d)\) is called 2-metric space.

Definition 3. A sequence \(\{x_n\}\) in 2-metric space \((X,d)\) is said to be convergent to an element \(x \in X\) if \(\lim_{n \to \infty} d(x_n,x,a) = 0\) for all \(a \in X\). It follows that if the sequence \(\{x_n\}\) converges to \(x\) then \(\lim_{n \to \infty} d(x_n,a,b) = d(x,a,b)\) for all \(a,b \in X\).
Definition 4. A sequence \( \{x_n\} \) in 2-metric space \((X,d)\) is said to be Cauchy sequence if \( d(x_m, x_n, a) = 0 \) as \( m,n \to \infty \) for all \( a \in X \).

Definition 5. A 2-metric space \((X,d)\) is said to be complete if every Cauchy sequence in \( X \) is convergent.

Proposition 6. If a sequence \( \{x_n\} \) in a 2-metric space converges to \( x \) then every subsequence of \( \{x_n\} \) also converges to the same limit \( x \).

Proposition 7. Limit of a sequence in a 2-metric space, if exists, is unique.

Definition 8. Let \( f \) and \( g \) be two self maps on a set \( X \). Maps \( f \) and \( g \) are said to be commuting if \( fgx = gfx \) for all \( x \in X \).

Definition 9. Let \((X,d)\) is a 2-metric space and \( f, g : (X,d) \to (X,d) \). The mappings \( f \) and \( g \) are said to be compatible if whenever \( \{x_n\} \) is a sequence in \( X \) such that \( \lim_{n \to \infty} fx_n = \lim_{n \to \infty} gx_n = t \) for some \( t \in X \) then \( d(fgx_n, gfx_n, a) \to 0 \) as \( n \to \infty \) for all \( a \in X \).

Definition 10. Let \( f \) and \( g \) be two self maps in a 2-metric space \((X,d)\) then \( f \) and \( g \) are said to be weakly compatible if they commute at their coincidence points.

Lemma 11. Let \( f \) and \( g \) be weakly compatible self mapping of a set \( X \). If \( f \) and \( g \) have a unique point of coincidence \( z \), then \( z \) is the unique common fixed point of \( f \) and \( g \).

Main Result

Theorem 12. Let \( S \) and \( T \) be self compatible maps of a complete 2-metric space \((X,d)\) satisfying the following conditions

\[(i) \quad S(X) \subseteq T(X) \quad (1)\]
\[(ii) \quad \psi \left( \int_0^{d(Sx, Sy, a)} \varphi(t) dt \right) \leq \psi \left( \int_0^{d(Tx, Ty, a)} \varphi(t) dt - \varphi \right) \int_0^{d(Tx, Ty, a)} \varphi(t) dt \quad (2)\]

for each \( x, y, a \in X \), where \( \psi : [0, \infty) \to [0, \infty) \) is a continuous and non decreasing function and \( \varphi : [0, \infty) \to [0, \infty) \) is a lower semi continuous and non decreasing function such that \( \psi(t) = \varphi(t) = 0 \) if and only if \( t = 0 \). Then \( S \) and \( T \) have a unique common fixed point.

Proof: Let \( x_0 \) be an arbitrary point of \( X \). Since \( S(X) \subseteq T(X) \), choose a point \( y_1 \) in \( X \) such that \( y_1 = Sy_0 = Ty_0 \). Continuing this process, in general, choose \( y_{n+1} = Ty_n \), \( n = 0, 1, 2, \ldots \). For each integer \( n \geq 1 \), and for all \( a \in X \), we have from (2)

\[ \psi \left( \int_0^{d(y_n, y_{n+1}, a)} \varphi(t) dt \right) \leq \psi \left( \int_0^{d(y_{n+1}, y_{n+2}, a)} \varphi(t) dt - \varphi \right) \int_0^{d(y_{n+1}, y_{n+2}, a)} \varphi(t) dt \]

\[ \leq \psi \left( \int_0^{d(y_{n+1}, y_{n+2}, a)} \varphi(t) dt \right) \quad (3) \]

Since \( \psi \) is continuous and has a monotone property. Therefore

\[ \int_0^{d(y_{n+1}, y_{n+2}, a)} \varphi(t) dt \leq \int_0^{d(y_n, y_{n+1}, a)} \varphi(t) dt \]
Let us take \( z_n = \int_{0}^{\infty} \phi(t)dt \) then it follows that \( z_n \) is monotone decreasing and lower bounded sequence of numbers. Therefore there exist \( k \geq 0 \) such that \( z_n \to k \) as \( n \to \infty \). Suppose that \( k > 0 \). Taking limit as \( n \to \infty \) on both sides of (3) and using that \( \phi \) is lower semi continuous, we get,

\[
\psi(k) \leq \psi(k) - \phi(k) < \psi(k)
\]

This is a contradiction. Therefore \( k = 0 \). This implies \( z_n \to 0 \) as \( n \to \infty \).

\[
\int_{0}^{\infty} \phi(t)dt \to 0
\]

Now we prove that \( \{y_n\} \) is a Cauchy sequence. Suppose it is not. Therefore there exists an \( \epsilon > 0 \) and subsequence \( \{y_{m(p)}\} \) and \( \{y_{n(p)}\} \) such that for each positive integer \( p(m), p(n) \) such that \( m(p) < n(p) < m(p + 1) \) with

\[
d(y_{m(p)}, y_{m(p)}, a) \geq \epsilon, \quad d(y_{n(p)}, y_{m(p)}, a) < \epsilon
\]

Now

\[
\epsilon \leq d(y_{n(p)}, y_{m(p)}, a) \leq d(y_{n(p)}, y_{m(p)}, y_{n(p) - 1}) + d(y_{n(p)}, y_{n(p) - 1}, a) + d(y_{n(p) - 1}, y_{m(p)}, a)
\]

\[
+ d(y_{n(p) - 1}, y_{m(p) - 1}, a) + d(y_{m(p) - 1}, y_{m(p)}, a)
\]

\[
\leq d(y_{n(p)}, y_{m(p)}, y_{n(p) - 1}) + d(y_{n(p)}, y_{n(p) - 1}, a) + d(y_{n(p) - 1}, y_{m(p)}, y_{m(p) - 1})
\]

\[
+ d(y_{n(p) - 1}, y_{m(p) - 1}, a) + d(y_{m(p) - 1}, y_{m(p)}, a) = \alpha
\]

\[
d(y_{n(p) - 1}, y_{m(p) - 1}, a) \leq d(y_{n(p) - 1}, y_{m(p) - 1}, y_{n(p)}) + d(y_{n(p) - 1}, y_{n(p)}, a) + d(y_{n(p)}, y_{m(p) - 1}, a)
\]

\[
d(y_{n(p) - 1}, y_{m(p) - 1}, a) \leq d(y_{n(p) - 1}, y_{m(p) - 1}, y_{n(p)}) + d(y_{n(p) - 1}, y_{n(p)}, a) + d(y_{n(p)}, y_{m(p) - 1}, y_{m(p)})
\]

\[
+ d(y_{n(p) - 1}, y_{m(p) - 1}, a) + d(y_{m(p) - 1}, y_{m(p)}, a) = \beta
\]

and therefore

\[
\int_{0}^{\infty} \phi(t)dt \leq \int_{0}^{\infty} \phi(t)dt
\]

\[
\int_{0}^{\infty} \phi(t)dt \leq \int_{0}^{\infty} \phi(t)dt
\]
Taking $\rho \rightarrow \infty$ and using (5) and (8) in (11) and (12), we get

\[ \int_0^\infty \varphi(t)dt \leq \int_0^\infty \varphi(t)dt \leq \int_0^\infty \varphi(t)dt \]

This implies,

\[ \lim_{\rho \rightarrow \infty} \int_0^\infty \varphi(t)dt = \ell \quad (13) \]

Now from (2), we have

\[ \psi \int_0^\infty \varphi(t)dt \leq \psi \int_0^\infty \varphi(t)dt - \phi \int_0^\infty \varphi(t)dt \quad (14) \]

Taking limit as $\rho \rightarrow \infty$ and using (8) and (13) in (14) we get

\[ \psi(\ell) \leq \psi(\ell) - \phi(\ell) \]

This is a contradiction. Hence \( \{y_n\} \) is a Cauchy sequence. Since \((X,d)\) is complete 2-metric space, therefore there exists a point \( v \) such that

\[ Sx_n \rightarrow v \& Tx_n \rightarrow v \] as \( n \rightarrow \infty \). Consequently, we can find \( h \) in \( X \) such that \( T(h) = v \).

Now, for all \( a \in X \) we have

\[ \psi \int_0^\infty \varphi(t)dt \leq \psi \int_0^\infty \varphi(t)dt - \phi \int_0^\infty \varphi(t)dt \]

On taking Limit as \( n \rightarrow \infty \) implies

\[ \psi(\int_0^\infty \varphi(t)dt) \leq \psi(0) - \varphi(0) \]

So

\[ \psi(\int_0^\infty \varphi(t)dt) = 0 \]

Hence \( v \) is the point of coincidence of \( S \) and \( T \).

Now we prove that \( v \) is the unique point of coincidence of \( S \) and \( T \). Suppose not, therefore there exists \( u \), \( (u \neq v) \) and there exists \( \mu \in X \) such that \( S \mu = T \mu = u \).

Using (2) we have for all \( a \in X \)

\[ \psi \int_0^\infty \varphi(t)dt = \psi \int_0^\infty \varphi(t)dt \leq \psi \int_0^\infty \varphi(t)dt - \phi \int_0^\infty \varphi(t)dt \]

Therefore

\[ \psi \int_0^\infty \varphi(t)dt < \psi \int_0^\infty \varphi(t)dt \]

This is a contradiction which implies \( u = v \). This proves uniqueness of point of coincidence of \( S \) and \( T \). Therefore by using lemma (11), the theorem is proved.

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References