Introduction

The line graph of a graph $G$, denoted by $L(G)$, is the graph whose vertices are the edges of $G$ with two vertices of $L(G)$ are adjacent whenever the corresponding edges of $G$ are adjacent.

The concept of pathos of a graph $G$ was introduced by Harary [1], as a collection of minimum number of edge disjoint open paths whose union is $G$. The path number of a graph $G$ is the number of paths in pathos. A Binary Tree $T$ is a tree in which each vertex has at most two children. The order (size) of $T$ is the number of vertices (edges) in it. The path number of a binary tree $T$ is equal to $\alpha$, where $2\alpha$ is the number of odd degree vertices of $T$. The edge degree of an edge $uv$ of a binary tree $T$ is the sum of the degrees of $u$ and $v$ [2]. A graph $G$ is planar if it can be drawn in the plane in such a way that any intersection of two distinct edges occurs only at a vertex of the graphs. A graph $G$ is called outerplanar if $G$ has an embedding in the plane in such a way that each vertex bounds the infinite face. An outerplanar graph $G$ is maximal outerplanar if no edge can be added without losing its outer planarity. If $G$ is a planar graph, then the inner vertex number $i(G)$ of a graph $G$ is the minimum number of vertices not belonging to the boundary of the exterior region in any embedding of $G$ in the plane [3]. A graph is said to be minimally non-outerplanar if $i(G)=1$[4]. All other undefined terminology will confirm with that in [5].

The pathos adjacency line graph $PAL(T)$ of a binary tree $T$ is defined as a graph, in which;

a) $V(PAL(T))$ is the union of the set of edges and paths of $T$ in which two vertices are adjacent if and only if the corresponding edges of $T$ are adjacent and edges lies on the corresponding path $p_i$ of pathos.

b) With reference to the binary tree $T$, the pathos vertex $p_m(v_i,v_j)$ is adjacent to $p_n(v_k,v_l)$ in $PAL(T)$ if and only if the pathos $p_m$ and $p_n$ have the common vertex $v_c$ in $T$ such that there exists an edge between $v_i$ and $v_j$ through $v_c$.

Since the system of pathos for a binary tree is not unique, the corresponding pathos adjacency line graph is also not unique. In the following figure, $T$ is a binary tree and its $PAL(T)$ is shown.

**Binary Tree T:**

**PAL(T):**

We need the following results to prove further results:
Theorem [A][2]: The pathos line graph $PL(T)$ of a tree $T$ is maximal outerplanar if and only if $T$ is a Path.

Theorem [B][5]: If $G$ is a $(p, q)$-graph whose vertices have degree $d_i$, then $L(G)$ has $q$ vertices and $q_L$ edges, where

$$q_L = -q + \frac{1}{2} \sum d_i^2.$$

Theorem [C][5]: A connected graph $G$ is Eulerian if and only if each vertex in $G$ has even degree.

Theorem [D][5]: A graph $G$ is outerplanar if and only if it has no subgraph homeomorphic to $K_4$ or $K_{2,3}$.

$PAL(T)$ Decomposition and Reconstruction of $T$:

We recall a graph a complete bipartite $(m,n)$-graph on $m+n$ vertices is the simple graph $G = K_{m,n}$ where

$$V(K_{m,n}) = \{u_1, u_2, \ldots, u_m\} \cup \{v_1, v_2, \ldots, v_n\},$$

$$E(K_{m,n}) = \{\{u_i, v_j\}: 1 \leq i \leq m, 1 \leq j \leq n\}.$$

that is, the vertices are of two types, where every pair of vertices of different types are adjacent, but no two vertices of the same type are adjacent. We have the following cases.

Case 1: Here we decompose $L(T)$ of $PAL(T)$ into an edge disjoint complete bipartite subgraphs as follows. Let

$$\{v_1, v_2, \ldots, v_n\}$$

are the vertices of subgraph $L(T)$ of $PAL(T)$. Then two vertices $v_i, v_j (v_i \neq v_j)$ are adjacent in $PAL(T)$ decomposition if they are adjacent in $L(T)$ such that they not form a cycle in $L(T)$ and the edge forming a cycle in $L(T)$ is taken as a separate component.

Case 2: The pathos vertex corresponding to pathos of $T$ and vertices corresponding to edges of $T$ are adjacent in $PAL(T)$ decomposition if they are adjacent in $PAL(T)$.

Case 3: Two pathos vertices $P_i, P_j (P_i \neq P_j)$ are adjacent in $PAL(T)$ decomposition if they are adjacent in $PAL(T)$. Hence, $PAL(T)$ consists of mutually edge disjoint complete bipartite subgraphs.

Conversely, let $T$ be the graph of the type which is described above. We can construct the binary tree $T$ which has $T$ as its pathos adjacency line graph as follows.

We first consider the line graph $L(T)$ decomposition of $PAL(T)$. Let the vertices $\{v'_1, v'_2, \ldots, v'_n\}$ of $L(T)$ decomposition are the edges in $T$. Then if two vertices $v'_i, v'_j (v'_i \neq v'_j)$ are adjacent in decomposition, then the corresponding edges $v_i, v_j$ are adjacent in $T$. We finally consider the $PAL(T)$ decomposition components $L_i$ having vertices corresponding to pathos and edges of $T$. Then, draw the pathos in $T$ along each of the pendant vertices corresponding to edges of $T$ in decomposition components.
Hence, $T$ is connected. If $\text{PAL}(T)$ is $K_2$, then $T$ is also $K_2$. Let $T$ be a connected binary tree with $p \geq 2$ vertices, $q$-edges and the path number $k$. Then, $\text{PAL}(T)$ has $(q+k)$ vertices. Since each edge in $T$ lies on exactly one path of pathos, $P-I=0$. Hence, by Theorem [2],

$$|\text{PAL}(T)| = \frac{1}{2} \sum_{i=1}^{p} d_i^2$$

Then, by Theorem [A], $\text{PAL}(T)$ is maximal outerplanar.

**Theorem 6:** Pathos adjacency line graph $\text{PAL}(T)$ of a binary tree $T$ is minimally non-outerplanar if and only if $T$ has unique vertex of degree 3.

**Proof:** Suppose $\text{PAL}(T)$ is minimally non-outerplanar. Assume that there exists at least two vertices of degree 3 in $T$. Then each block in $L(T)$ is either $K_2$ or $K_3$. Any pathos vertex of $T$ is adjacent to at most two vertices of each block of $L(T)$. Also, the adjacency of pathos vertices gives $\text{PAL}(T)$ such that $i(\text{PAL}(T)) > 1$, a contradiction.

Conversely, suppose $T$ has a unique vertex of degree three. Then $L(T)$ has only one block as $K_3$. Also, the number of path of pathos in $T$ is exactly two. Each pathos vertex is adjacent to at most two vertices of $K_3$. Finally, the adjacency of pathos vertices gives $\text{PAL}(T)$ such that $i(\text{PAL}(T)) = 1$. Hence, $\text{PAL}(T)$ is minimally non-outerplanar.

**Theorem 7:** Pathos adjacency line graph $\text{PAL}(T)$ of a binary tree $T$ is Eulerian if and only if $T$ is a path $P_n$ on $n \geq 4$ vertices. Then it forms a path of length $(n-2)$ in $L(T)$ in which degree of each vertex except end vertices is two. Also, $T$ has exactly one path of pathos. The edges joining vertices of $L(T)$ from the pathos vertex increases the degree of even vertices of $L(T)$ by one in $\text{PAL}(T)$.

By Theorem [C], $\text{PAL}(T)$ is non-Eulerian, a contradiction. Conversely, suppose $T$ is a path $P_n$ on $n=3$ vertices. Then each edge in $T$ lies on exactly one pathos. By the definition of $L(T)$ it forms $K_2$. The edges joining vertices of $K_2$ from the pathos vertex forms $K_3$ in $\text{PAL}(T)$. Then, by Theorem [C], $\text{PAL}(T)$ is Eulerian.

**References:**


