1. Introduction and statement of results

In the theory of distribution of zeros of polynomials, the Enestrom-Kakeya theorem [4] given below in theorem A is a well known result.

**Theorem A.** Let \( P(z) = \sum_{i=0}^{n} a_i z^i \) be a polynomial of degree \( n \) such that

\[
a_n \geq a_{n-1} \geq \ldots \geq a_1 \geq a_0 > 0
\]

Then all the zeros of \( P(z) \) lie in the disk \( |z| \leq 1 \).

Many attempts have been made to extend and generalize the Enestrom-Kakeya theorem. A. Joyal et al [3] extended the Enestrom-Kakeya theorem to the polynomials with general monotonic coefficients by proving that if

\[
a_n \geq a_{n-1} \geq \ldots \geq a_1 \geq a_0
\]

Then \( P(z) \) has all its zeros in the disk

\[
|z| \leq \frac{a_n - a_0 + |a_0|}{|a_n|}
\]

Further Aziz and Zargar [1] generalized the result of A. Joyal et al [3] and the Enestrom-Kakeya theorem as given below in theorem B.

**Theorem B.** Suppose \( P(z) = \sum_{i=0}^{n} a_i z^i \) be a polynomial of degree \( n \) such that

For some \( \lambda \geq 1 \),

\[
\lambda a_n \geq a_{n-1} \geq \ldots \geq a_1 \geq a_0
\]

then all the zeros of \( P(z) = \sum_{i=0}^{n} a_i z^i \), lie in the disk

\[
|z + (\lambda - 1)| \leq \frac{\lambda a_n - a_0 + |a_0|}{|a_n|}
\]
Theorem C. Suppose \( P(z) = \sum_{i=0}^{n} a_i z^i \) be a polynomial of degree \( n \) such that either
\[
a_n \geq a_{n-2} \geq \ldots \geq a_2 \geq a_0 > 0 \quad \text{and} \quad a_{n-1} \geq a_{n-3} \geq \ldots \geq a_3 \geq a_1 > 0 \quad \text{if} \quad n \text{ is even}
\]
Or,
\[
a_n \geq a_{n-2} \geq \ldots \geq a_2 \geq a_0 > 0 \quad \text{and} \quad a_{n-1} \geq a_{n-3} \geq \ldots \geq a_2 \geq a_1 > 0 \quad \text{if} \quad n \text{ is odd, then all}
\]
the zeros of \( P(z) \) lie in the disk
\[
|z + \frac{a_{n-1}}{a_n}| \leq \frac{a_{n-1}}{a_n} \sum_{i=0}^{n} a_i z^i
\]
But Govil and Rahman [2] proved that if, \( P(z) = \sum_{i=0}^{n} a_i z^i \) is a complex polynomial of degree \( n \) with
\[
|\arg a_i - \beta| \leq \frac{\pi}{2}, \quad (i = 0, 1, 2, \ldots, n) \quad \text{for some} \quad \beta \quad \text{real and}
\]
\[
|a_n| \geq |a_{n-1}| \geq \ldots \geq |a_1| \geq |a_0|, \quad \text{then} \quad P(z) \quad \text{has all its zeros in the disk}
\]
\[
|z| \leq \cos \alpha + \sin \alpha + \frac{2 \sin \alpha}{|a_n|} \sum_{i=0}^{n-1} |a_i|
\]

Theorem D. Suppose \( P(z) = \sum_{i=0}^{n} a_i z^i \) be a complex polynomial of degree \( n \) with \( \Re(a_i) = \alpha_i \) and \( \Im(a_i) = \beta_i \), \( i = 0, 1, 2, \ldots, n \). If for some \( \lambda \geq 1 \)
\[
\lambda a_n \geq \lambda a_{n-1} \geq \ldots \geq \lambda a_1 \geq \lambda a_0, \quad \beta_n \geq \beta_{n-1} \geq \ldots \geq \beta_1 \geq \beta_0 > 0
\]
Then \( P(z) \) has all its zeros in the disk
\[
|z + \left( \frac{1}{\lambda^2} \lambda a_n - a_0 + |a_0| + \beta_0 \right)| \leq \sum_{i=0}^{n} a_i z^i
\]

Theorem E. Suppose \( P(z) = \sum_{i=0}^{n} a_i z^i \) be a complex polynomial of degree \( n \) such that
\[
|\arg a_i - \beta| \leq \frac{\pi}{2}, \quad (i = 0, 1, 2, \ldots, n) \quad \text{for some} \quad \beta \quad \text{real and for some} \quad \lambda \geq 1
\]
\[
\lambda |a_n| \geq |a_{n-1}| \geq \ldots \geq |a_1| \geq |a_0|
\]
Then \( P(z) \) has all its zeros in the disk
\[
|z + (\lambda - 1)| \leq \frac{1}{|a_n|} \{ (\lambda |a_n| - |a_0|) (\cos \alpha + \sin \alpha) + |a_0| + \sin \alpha \sum_{i=0}^{n-1} |a_i| \}
\]

The main purpose of this paper is to refine some results mentioned above and define the zero-free regions of polynomials in theorems C, D and E.

2. Theorems And Proofs

Theorem 1. Suppose \( P(z) = \sum_{i=0}^{n} a_i z^i \) be a polynomial of degree \( n \) such that either
\[
a_n \geq a_{n-2} \geq \ldots \geq a_2 \geq a_0 \quad \text{and} \quad a_{n-1} \geq a_{n-3} \geq \ldots \geq a_3 \geq a_1 \quad \text{if} \quad n \text{ is even}
\]
Or, \( a_n \geq a_{n-2} \geq \ldots \geq a_2 \geq a_0 \) and \( a_{n-1} \geq a_{n-3} \geq \ldots \geq a_3 \geq a_1 \) if \( n \) is odd
Then \( P(z) \) does not vanish in the disk
\[
|z| \leq \frac{|a_0|}{|a_n| + |a_{n-1}| + |a_{n-2}| + |a_{n-3}| + |a_1| - a_0}
\]

Proof. To prove the theorem, we consider a polynomial \( F(z) \) defined by
From above, $g(0) = 0$, therefore by Schwarz lemma, it follows that $M = |a_0|$.

On simplification, we have

$$|g(z)| \leq |a_0| + |a_1(n-1)| + (a_1n - a_1(n-2)) + (a_{n-1} - a_1(n-3)) + \ldots + (a_3 - a_1) + (a_2 - a_0)| + |a_1|$$

since by hypothesis $a_n \geq a_{n-2} \geq \ldots \geq a_2 \geq a_0$. Therefore $a_n \geq a_{n-2} \geq \ldots \geq a_2 \geq a_1$.

Theorem 2. Let $P(z) = \sum_{i=0}^{n} a_i z^i$ be a complex polynomial of degree $n$ with

$$\text{Re}(a_i) = \alpha_i \quad \text{and} \quad \text{Im}(a_i) = \beta_i, \quad i=0,1,2,\ldots,n.$$ 

Let for some $\lambda \geq 1$,

$$\lambda \alpha_n \geq \alpha_{n-1} \geq \ldots \geq \alpha_1 \geq \alpha_0,$$

$$\beta_n \geq \beta_{n-1} \geq \ldots \geq \beta_1 \geq \beta_0,$$

then $P(z)$ does not vanish in the disk $|z| < 1$.

Proof. To prove the theorem, we consider a polynomial $F(z)$ defined by

$$F(z) = (1 - z) \cdot P(z) = (1 - z)^n \sum_{i=0}^{n} a_i z^i$$

On simplification, we have

$$F(z) = -a_n z^{n-1} + (a_n - a_1(n-1)) z^{n-2} + \ldots + (a_1 - a_0) z + a_0$$

Using hypothesis, we can write $g(z) = a_0$ where

$$g(z) = -a_n z^{n-1} + (a_n - a_1(n-1)) z^{n-2} + \ldots + (a_1 - a_0) z + a_0$$

Now if $|z| < 1$, then on simplification, we have

$$|g(z)| \leq |a_0| + (\lambda - 1)|a_n| + \lambda |a_n - a_0| + \beta_n - \beta_0$$

From above, $g(0) = 0$, therefore by Schwarz lemma, it follows that...
\[ |g(z)| \leq M|z| \quad \text{for} \quad |z| < 1 \quad , \text{where} \quad M = |a_n| + (\lambda - 1)|\alpha_n| + \lambda \alpha_n - \alpha_0 + \beta_n - \beta_0 \]

Again for \(|z| < 1\),
\[ |F(z)| = |g(z) + a_0| \geq |a_o| - |g(z)| \geq |a_o| - M|z| > 0, \quad \text{if} \quad |a_o| > M|z| \]

i.e., if \[ |z| < \frac{|a_o|}{M} \]

where \[ M = |a_n| + (\lambda - 1)|\alpha_n| + \lambda \alpha_n - \alpha_0 + \beta_n - \beta_0 \]

Also we can show that \[ M \geq |a_o| \quad \text{as} \quad |z| < 1 \]

Hence the desired result follows.

**Theorem 3.** Suppose \(P(z) = \sum_{i=0}^{n} a_i z^i\) be a complex polynomial of degree \(n\) such that \[ |\arg a_i - \beta| \leq \alpha \leq \frac{\pi}{2} \quad , \quad (i = 0, 1, 2, \ldots, n) \]

for some \(\beta\) real and for some \(\lambda \geq 1\)

Then \(P(z)\) does not vanish in the disk \(\{ z : |z| < \frac{|a_o|}{\lambda |a_n| - |a_o|} \cos \alpha + \left( |\lambda a_n| + |a_o| \right) \sin \alpha + 2 \sin \alpha \sum_{i=0}^{n-1} |a_i| \}\)

**Proof.** To prove the theorem, we consider a polynomial \(F(z)\) defined by
\[
F(z) = (1 - z) P(z) = (1 - z) \left( a_0 + a_1 z + a_2 z^2 + \ldots + a_{n-1} z^{n-1} + a_n z^n \right) \\
= -a_n z^{n+1} + (a_n - a_d(n-1)) z^n + \ldots + (a_1 - a_0) z + a_0 \\
= -a_n z^{n+1} - (\lambda - 1) a_n z^n + (\lambda a_d n - a_d(n-1)) z^{n+1} + \ldots + (a_1 - a_0) z + a_0 \\
= g(z) + a_0, \quad \text{where} \quad g(z) = -a_n z^{n+1} - (\lambda - 1) a_n z^n + (\lambda a_d n - a_d(n-1)) z^{n+1} + \ldots + (a_1 - a_0) z \\
\]

It was shown in [2] that for two complex numbers \(b_o, b_1\) if
\[ |b_o| \geq |b_1| \quad \text{and} \quad |\arg b_i - \beta| \leq \alpha \leq \frac{\pi}{2} , \quad (i = 0, 1) \]

for some \(\beta\) then
\[ |b_o - b_1| \leq (|b_o| - |b_1|) \cos \alpha + (|b_o| + |b_1|) \sin \alpha \]

Hence for \(|z| < 1\),
\[ |g(z)| \leq |a_n| + (\lambda - 1)|\alpha_n| + (|\lambda a_n| - |a_0|) \cos \alpha + (|\lambda a_n| + |a_0|) \sin \alpha + 2 \sin \alpha \sum_{i=0}^{n-1} |a_i| \]

Again we have, \(g(0) = 0\), therefore by Schwarz lemma we obtain
\[ |g(z)| \leq M|z| \quad \text{for} \quad |z| < 1 \quad , \text{where} \quad M = \lambda |a_n| + (\lambda |a_n| - |a_0|) \cos \alpha + (|\lambda a_n| + |a_0|) \sin \alpha + 2 \sin \alpha \sum_{i=0}^{n-1} |a_i| \]

Therefore for \(|z| < 1\), we have
\[ |F(z)| = |g(z) + a_0| \geq |a_o| - |g(z)| \geq |a_o| - M|z| > 0, \quad \text{if} \quad |a_o| > M|z| \]

i.e., if \[ |z| < \frac{|a_o|}{M} \]

\[ M = \lambda |a_n| + (\lambda |a_n| - |a_0|) \cos \alpha + (|\lambda a_n| + |a_0|) \sin \alpha + 2 \sin \alpha \sum_{i=0}^{n-1} |a_i| \]

Also we can show that \( M \geq |a_o| \quad \text{as} \quad |z| < 1 \)
Hence the desired result follows.

References