Cubic graphs with equal complementary tree domination number and chromatic number

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Abstract

A set D of vertices in a graph G = (V, E) is called a dominating set of G, if every vertex in V-D is adjacent to some vertex in D. The domination number \(\gamma(G)\) of G is the minimum cardinality of a dominating set in G. A dominating set D of a graph G is a complementary tree dominating set, if the induced subgraph \(<V-D>\) is a tree. The minimum cardinality of a complementary tree dominating set is called the complementary tree domination number of G and is denoted by \(\gamma_{ctd}(G)\). The chromatic number of G is the minimum number of colours assigned to the vertices of G such that no two adjacent vertices have the same colour and is denoted by \(\chi(G)\). In this paper we investigate cubic graphs for which \(\gamma_{ctd}(G) = \chi(G)\).

1. Introduction

Graphs considered here are nontrivial, simple, finite, connected and undirected. Let G be a graph with vertex set V(G) and edge set E(G). A graph with p vertices and q edges is denoted by G(p, q). In general, we use \(<X>\) to denote the subgraph induced by the set X of vertices. The concept of domination was first studied by Ore [4]. A set D \(\subseteq\) V is called a dominating set of G, if every vertex in V-D is adjacent to some vertex in D. A dominating set D of a graph G is a complementary tree dominating set, if every vertex in V-D is adjacent to some vertex in D and D is said to be a minimal dominating set if D-{u} is not a dominating set for any u \(\in\) D. The domination number \(\gamma\) of G is the minimum cardinality of a dominating set.

The concept of complementary tree domination was introduced by Muthammai, Bhanumathi and Vidhya[3]. A dominating set \(D\subseteq V\) is called a complementary tree dominating set \((ctd-set)\), if the sub graph \(<V-D>\) induced by \(V-D\) is a tree. The minimum cardinality of a complementary tree dominating set is called the complementary tree domination number of G and is denoted by \(\gamma_{ctd}(G)\). We call a set of vertices a \(\gamma\)-set, if it is a dominating set with cardinality \(\gamma\) of G. Similarly a \(\gamma_{ctd}\)-set is defined. A colouring of a graph is an assignment of colours to its vertices so that no two adjacent vertices have the same colour. An n-colouring of a graph G uses n colours. The chromatic Number \(\chi\) of G is defined to be the minimum n, for which G has an n-colouring.

Several authors have studied the problem of obtaining an upper bound for the sum of a domination parameter and a graph theoretic parameter and characterized the corresponding extremal graphs. There are several papers in which graphs with equal parameters are investigated. In[5], Paulraj Joseph investigated cubic graphs whose domination number equals chromatic number. Motivated by the above, we now took the problem of characterizing the graphs for which complementary tree domination number equals to chromatic number. For terminology and notations not specifically defined here we refer reader to [1]. For more details about domination number and its related parameters, we refer to [2].

In this paper, we investigate cubic graphs whose complementary tree domination number equals chromatic number.

2. Prior Results

Theorem 2.1. [1] If G is neither a complete graph nor an odd cycle, then \(\chi(G) \leq \Delta(G)\).

Theorem 2.2. [1] For any connected graph G, \(\chi(G) \leq \Delta(G) + 1\).

Theorem 2.3. [2]. If G is a graph of order p, with maximum degree \(\Delta\), then \(\gamma(G) \leq \left\lfloor \frac{p}{\Delta+1} \right\rfloor\).

Theorem 2.4. [5]. For a connected graph G, \(\chi(G) = \chi(G) = 2\) if and only if

(i) G is bipartite with bipartition (X, Y) and

(ii) \(|X| = 2\) (or)

there exist x in X, y in Y such that N(x) = Y and N(y) = X (or)

there exist x in X and y in Y such that N(x) = Y \(\setminus\{y\}\) and N(y) = X \(\setminus\{x\}\).

3. Main Results

In the following we find the cubic graphs with equal complementary tree domination number and chromatic number. In analogous to Theorem 2.4., we state the following theorem.

Theorem 3.1. For a connected graph G, with at least 3 vertices, \(\gamma_{ctd}(G) = \chi(G) = 2\) if and only if G is bipartite with bipartition (X, Y) such that there exist x in X, y in Y with V(G) \(\setminus\{x, y\}\) is a tree and either N(x) = Y and N(y) = X (or) N(x) = Y \(\setminus\{y\}\) and N(y) = X \(\setminus\{x\}\).
Let $G = (V, E)$ be a connected cubic graph of order $p$ with $\gamma_{ctd}(G) = \chi(G)$.

But $\chi(G) \leq \Delta(G)$ implies $\chi(G) \leq 3$. Clearly $\chi(G) \neq 1$.

Theorem 3.1. characterizes graphs for which $\gamma_{ctd}(G) = \chi(G) = 2$. We consider the cubic graphs for which $\gamma_{ctd}(G) = \chi(G) = 3$.

Then $\gamma_{ctd}(G) \geq \frac{p}{\Delta + 1} = \frac{p}{4}$. Since $\gamma_{ctd}(G) = 3$, $6 < p \leq 15$ and $p \neq 14$. Also, $p$ is even, since $G$ is cubic. Therefore, $p = 8, 10$ (or) 12.

Cubic Graphs of Order 8

Theorem 3.2.

Let $G$ be a connected cubic graph on 8 vertices. Then $\gamma_{ctd}(G) = \chi(G) = 3$, if and only if $G$ is isomorphic to any of the six graphs $G_1, G_2, G_3, G_4$ and $G_5$ given in Figure 1.

Proof:

Let $G$ be a connected cubic graph on 8 vertices such that $\gamma_{ctd}(G) = \chi(G) = 3$.

Let $D = \{u, v, w\}$ be a minimum ctd-set of $G$ and let $V - D = \{x_1, x_2, x_3, x_4, x_5\}$. It is clear that $\langle D \rangle \neq K_3$. If $\langle D \rangle \equiv K_3$, then there exist at least two vertices in $V - D$ not adjacent to any of the vertices in $D$. Since $(V - D)$ is a tree, either $(V - D)$ is a path on 5 vertices or $\langle V - D \rangle \equiv T$, where $T$ is the tree obtained from the path on four vertices by attaching a pendant edge at any one of the supports. Second case is not possible, since there is a vertex of degree 3 in $(V - D)$ and it cannot be adjacent to any of the vertices in $D$. Therefore, $(V - D)$ is a path on 5 vertices. Let $x_1, x_2, x_3, x_4, x_5$ be the vertices of $(V - D)$ and let $E((V - D)) = \{(x_1, x_2), (x_2, x_3), (x_3, x_4), (x_4, x_5)\}$.

Case 1. $\langle D \rangle \equiv K_3$

Let $u$ be adjacent to $x_1, x_2, x_3$. Then $v$ is adjacent to $x_1, x_3, x_4$ and $w$ is adjacent to $x_5$ only. Hence $\deg_G(w) = 1$, which is not possible. Similarly, if $u$ is adjacent to $x_2, x_3, x_4$ (or) $x_1, x_2, x_4$, then at least one of the vertices $v$ and $w$ has degree less than $3$ in $G$. Therefore, $\langle D \rangle \neq K_3$.

Case 2. $\langle D \rangle \equiv K_2 \cup K_1$

Let the vertices of $K_2$ be $u, v$ and let $w$ be the vertex of $K_1$. Subcase(a). $u$ is adjacent to $x_1, x_2$.

Then $\gamma_{ctd}(G) = \chi(G) = 3$ in the graphs $G_1, G_2$ and $G_3$ given in Figure 1.

N(v) – {u} N(w) G is isomorphic to

<table>
<thead>
<tr>
<th>$\chi(G)$</th>
<th>$\gamma_{ctd}(G)$</th>
<th>$G$ is isomorphic to</th>
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For the graphs $G_i$, $i = 1, 2, 3, 4, 5$, $\chi(G_i) = 3$ and for the graph $G_6$, $\chi(G_6) = 2$.

Therefore, $\gamma_{ctd}(G) = \chi(G) = 3$ if and only if $G \equiv G_i$, $i = 1, 2, 3, 4, 5$.

Conversely if $G \equiv G_i$, $i = 1, 2, 3, 4, 5$, then $\gamma_{ctd}(G) = \chi(G) = 3$.

Cubic Graphs of Order 10

Theorem 3.3.

Let $G$ be a connected cubic graph of order 10. Then $\gamma_{ctd}(G) = \chi(G) = 3$ if and only if $G$ is isomorphic to one of the seventeen graph $G_i$, $i = 1, 2, ..., 17$ given in Figure 2.

Proof:

Let $D$ be a $\gamma_{ctd}$-set of $G$ such that $|D| = 3$. Then $(V - D)$ is a tree.

Let $D = \{u, v, w\}$ and $V - D = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$.

Since the graph $G$ is cubic, $\langle V - D \rangle$ must be a path $P_7$ on 7 vertices. Let $x_1, x_7$ be the pendant vertices of $P_7$. Clearly, $\langle D \rangle \neq K_2$ and $P_7$. Hence, $\langle D \rangle \equiv K_2 \cup K_1$ (or) $K_3$.

If $\langle D \rangle \equiv K_2 \cup K_1$, then there exist at least two vertices in $G$ of degree at most two and hence $\langle D \rangle \neq K_2 \cup K_1$. Therefore, $\langle D \rangle \equiv K_3$.

Then the following two cases arise

(i) Two vertices of $D$ are adjacent to the both pendant vertices $x_1$ and $x_7$.
(ii) One vertex of $D$ is adjacent to both the pendant vertices $x_1$ and $x_7$ and each of the remaining two vertices of $D$ are adjacent to exactly one pendant vertex (distinct).
Case (1): u and v are adjacent to both $x_1$ and $x_2$.
Then w is adjacent to none of $x_1$ and $x_2$.
Let $u$ be adjacent to $x_1, x_7, x_2$. Then $v$ is adjacent to $x_1, x_7, x_3; x_1, x_7, x_4; x_1, x_7, x_5; x_1, x_7, x_6$.
Correspondingly, w is adjacent to $x_4, x_5, x_6; x_3, x_5, x_6; x_3, x_4, x_6$.
Then $G$ is isomorphic to one of the graphs $G_5, G_6, G_7, G_8, G_9, G_{10}$.

Case (2): u is adjacent to both $x_1$ and $x_7, v$ is adjacent to $x_1$ and w is adjacent to $x_7$.
Subcase (a): u is adjacent to $x_1, x_2, x_7$.
Then $v$ is adjacent to $x_1, x_3, x_4; x_1, x_3, x_5; x_1, x_3, x_6; x_1, x_4, x_5; x_1, x_4, x_6, x_5, x_6$.
Correspondingly, $w$ is adjacent to $x_3, x_6, x_5; x_4, x_5, x_6; x_3, x_4, x_6; x_3, x_5, x_6$.
Then $G$ is isomorphic to one of the graphs $G_{12}, G_{13}, G_{14}, G_{15}$.

Subcase (b): u is adjacent to $x_1, x_5, x_7$.
Then $v$ is adjacent to $x_1, x_2, x_3, x_4, x_5; x_1, x_2, x_3, x_4, x_6; x_1, x_2, x_3, x_5, x_6$.
Correspondingly, $w$ is adjacent to $x_3, x_4, x_6; x_3, x_4, x_5; x_3, x_5, x_6; x_3, x_4, x_5, x_6$.
Then $G$ is isomorphic to one of the graphs $G_{10}, G_{11}, G_{12}, G_{13}$.

Subcase (c): $u$ is adjacent to $x_1, x_4, x_7$.
Then $v$ is adjacent to $x_1, x_2, x_3, x_4, x_7; x_1, x_2, x_3, x_4, x_5, x_6$.
Correspondingly, $w$ is adjacent to $x_2, x_4, x_5, x_6; x_2, x_3, x_4, x_5, x_6$.
Then $G$ is isomorphic to one of the graphs $G_{10}, G_{11}, G_{12}, G_{13}$.

Subcase (d): $u$ is adjacent to $x_1, x_4, x_7$.
Then $v$ is adjacent to $x_1, x_2, x_3, x_4, x_7; x_1, x_2, x_3, x_4, x_5, x_6$.
Correspondingly, $w$ is adjacent to $x_2, x_4, x_5, x_6; x_2, x_3, x_4, x_5, x_6$.
Then $G$ is isomorphic to one of the graphs $G_{10}, G_{11}, G_{12}, G_{13}$.

Subcase (e): $u$ is adjacent to $x_1, x_4, x_7$.
Then $v$ is adjacent to $x_1, x_2, x_3, x_4, x_7; x_1, x_2, x_3, x_4, x_5, x_6$.
Correspondingly, $w$ is adjacent to $x_2, x_4, x_5, x_6; x_2, x_3, x_4, x_5, x_6$.
Then $G$ is isomorphic to one of the graphs $G_{10}, G_{11}, G_{12}, G_{13}$.

From the above cases, $G \cong G_i, i = 1, 2, ..., 17$ and for these graphs $\chi(G) = 3$.
Conversely, for the graphs $G_i, i = 1, 2, ..., 17, \gamma_{cd}(G) = \chi(G) = 3$. 

Figure 2
Cubic Graphs of Order 12

Theorem 3.4.

There exists no cubic graphs of order 12 such that $\gamma_{ctd}(G) = \chi(G) = 3$.

Proof:

Let G be a connected cubic graph of order 12, such that $\gamma_{ctd}(G) = \chi(G) = 3$.

Let D be a $\gamma_{ctd}$-set of G. Then $|D| = 3$ and $\langle V - D \rangle$ is a tree. Since each vertex in $V - D$ is adjacent to at least one vertex in D, each vertex in $V - D$ has degree either one or two in $\langle V - D \rangle$. That implies $\langle V - D \rangle$ is a path on nine vertices. Also, G has at least one vertex of degree less than or equal to two, which is a contradiction, since G is a cubic graph. Therefore, there exists no cubic graphs of order 12 having $\gamma_{ctd}(G) = \chi(G) = 3$.

Thus, we have found the cubic graphs of order 8 and 10 for which $\gamma_{ctd}(G) = \chi(G) = 3$.

References