On \(\pi gb\)-Separation Axioms in Bitopological Spaces

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ABSTRACT

In this paper, we introduce and study some new separation axioms using the \((1, 2)^\pi gb\)-open sets in bitopological spaces.
Definition 2.9: A bitopological space $X$ is $(1,2)^*\text{-}T_0$ if for each pair of distinct points $x, y$ of $X$, there exists a $(1,2)^*\text{-open}$ set containing one of the points but not the other. Complement of $(1,2)^*\text{-b-open}$ is called $(1,2)^*\text{-b-closed}$. Throughout the following sections by $X$ and $Y$ we mean bitopological spaces $(X, \tau_1, \tau_2)$ and $(Y, \sigma_1, \sigma_2)$ respectively.

$(1,2)^*\text{-gb-T}_0$-Spaces

Definition 3.1: A bitopological space $X$ is $(1,2)^*\text{-gb-T}_0$ if for each pair of distinct points $x, y$ of $X$, there exists a $(1,2)^*\text{-gb-open}$ set containing one of the points but not the other.

Lemma 3.2: If for some $x \in X$, $\{x\} \subseteq (1,2)^*\text{-gb-open}$, then $x \notin \{1,2\}^*\text{-gb-cl}([y])$ for all $y \neq x$.

Proof: If $\{x\} \subseteq (1,2)^*\text{-gb-open}$ for some $x \in X$, then $x \notin \{x\}$. If $x \in \{x\}^*\text{-gb-cl}([y])$ for some $y \neq x$, then $x,y$ both are in all the $(1,2)^*\text{-gb-closed}$ sets containing $y$. This implies $x \notin X$ which is not true. Hence $x \notin \{1,2\}^*\text{-gb-cl}([y])$.

Theorem 33: In a space $X$, distinct points have distinct $(1,2)^*\text{-gb-closures}$.

Proof: Let $x,y \in X$. $x \neq y$. Take $A = \{x\}^*$. Case(i): If $\tau_1 \tau_2\text{-cl}(A) = A$. Then $A \subseteq \tau_2\text{-closed}$. This implies $A \subseteq \{1,2\}^*\text{-gb-closed}$. Then $X-A = \{x\} \subseteq \{1,2\}^*\text{-gb-closed}$, not containing $y$. Then by previous lemma 3.2, $x \notin \{1,2\}^*\text{-gb-cl}([y])$ and $y \in \{1,2\}^*\text{-gb-cl}([y])$. Thus $(1,2)^*\text{-gb-cl}([x])$ and $(1,2)^*\text{-gb-cl}([y])$ are distinct.

Case(ii): If $\tau_1 \tau_2\text{-cl}(A) = X$. Then $A \subseteq \{1,2\}^*\text{-gb-open}$ and $\{x\} \subseteq \{1,2\}^*\text{-gb-closed}$. This implies $(1,2)^*\text{-gb-cl}([x]) = \{x\}$ which is not equal to $(1,2)^*\text{-gb-cl}([y])$.

Theorem 3.4: A bitopological space $X$ is $(1,2)^*\text{-gb-cl}(-T_0)$ iff for each pair of distinct points $x, y$ of $X$, $(1,2)^*\text{-gb-cl}(x) \neq (1,2)^*\text{-gb-cl}(y)$.

Proof: Necessity: Let $x \neq y$. Then there exists a $(1,2)^*\text{-gb-open}$ set $V$ containing one of the points but not the other, say $x \in V$ and $y \notin V$. Then $V^c$ is $(1,2)^*\text{-gb-closed}$ set containing $y$ but not $x$. But $(1,2)^*\text{-gb-cl}(y)$ is the smallest $(1,2)^*\text{-gb-closed}$ set containing $y$. Therefore $(1,2)^*\text{-gb-cl}(y) \subseteq V^c$ and hence $x \in (1,2)^*\text{-gb-cl}(y)$. Thus $(1,2)^*\text{-gb-cl}(x) \neq (1,2)^*\text{-gb-cl}(y)$.

Sufficiency: Suppose $x \neq y$. Then there exists a $(1,2)^*\text{-gb-open}$ set $V$ containing one of the points but not the other, say $x \in V$ and $y \notin V$. Then $V^c$ is $(1,2)^*\text{-gb-closed}$ set containing $y$ but not $x$. But $(1,2)^*\text{-gb-cl}(y)$ is the smallest $(1,2)^*\text{-gb-closed}$ set containing $y$. Therefore $(1,2)^*\text{-gb-cl}(y) \subseteq V^c$ and hence $x \in (1,2)^*\text{-gb-cl}(y)$. This is a contradiction. Hence $x \notin (1,2)^*\text{-gb-cl}(y)$.

Theorem 3.5: Every bitopological space is $(1,2)^*\text{-gb-T}_0$.

Proof: Follows from previous theorems 3.3 and 3.4.

Theorem 3.6: Let $f:X \to Y$ be a bijection, $(1,2)^*\text{-gb-open}$ map and $X$ is $(1,2)^*\text{-gb-T}_0$ space, then $Y$ is also $(1,2)^*\text{-gb-T}_0$ space.

Proof: Let $y_1, y_2 \in Y$ with $y_1 \neq y_2$. Since $f$ is a bijection, there exists $x_1,x_2 \in X$ with $x_1 \neq x_2$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Since $X$ is $(1,2)^*\text{-gb-T}_0$ space, there exists a $(1,2)^*\text{-gb-open}$ set $M$ in $X$ such that $x_1 \in M$ and $x_2 \notin M$. Since $f$ is $(1,2)^*\text{-gb-open}$, $f(M)$ is a $(1,2)^*\text{-gb-open}$ set in $Y$. Hence for any two distinct points $y_1, y_2$ in $Y$, there exists $(1,2)^*\text{-gb-open}$ set $f(M)$ in $Y$ such that $y_1 \notin f(M)$ and $y_2 \notin f(M)$. Hence $Y$ is $(1,2)^*\text{-gb-T}_0$ space.
Conversely, Suppose that \( \{ x \} \subset \cup (1,2)^{*}-gbO(X) \) but \((1,2)^{*}-gb-\text{cl}(\{ y \})\). This implies \((1,2)^{*}-\text{gb-cl}(\{ y \})\). Hence the set \( \{ y \} \) is a subset of \((1,2)^{*}-\text{gb-cl}(\{ x \})\). This \((1,2)^{*}-\text{gb-cl}(\{ y \})\) is a subset of \((1,2)^{*}-\text{gb-cl}(\{ x \})\). Now \((1,2)^{*}-\text{gb-cl}(\{ x \})\) contains \( x \) and \((1,2)^{*}-\text{gb-cl}(\{ y \})\) which is a contradiction.

**Remark 3.17:** Every \((1,2)^{*}-gb-T_{4}\) space is \((1,2)^{*}-gb-T_{0}\) space.

**Theorem 3.14:** In a space \( X \), the following are equivalent
\( \pi \) \( X \) is \((1,2)^{*}-gb-T_{1}\)
1. For every \( x \in X \), \( \{ x \} \) is \((1,2)^{*}-gb-T_{1}\).
2. Each subset \( A \) of \( X \) is the intersection of all \((1,2)^{*}-gb-T_{1}\) subsets containing \( A \).
3. The intersection of all \((1,2)^{*}-gb-T_{1}\) subsets containing the point \( x \) in \( X \) is \( \{ x \} \).

**Proof:** (1) \( \Rightarrow \) (2) Suppose \( X \) is \((1,2)^{*}-gb-T_{1}\). Let \( x \in X \) and \( y \in \{ x \} \). Then \( x \neq y \) and there exists a \((1,2)^{*}-gb-T_{1}\) subset \( U \) such that \( y \notin U \). Therefore \( y \notin U \subset \{ x \} \). That is, \( x \) is \((1,2)^{*}-gb-cl\) closed.

(2) \( \Rightarrow \) (3) Let \( A \subset X \) and \( y \notin A \). Then \( \{ y \} \) is \((1,2)^{*}-gb\)-open in \( X \) and \( \{ y \} \) is \((1,2)^{*}-gb\)-closed in \( \{ x \} \). Hence \( \{ x \} \) is \((1,2)^{*}-gb\)-closed.

(3) \( \Rightarrow \) (4) is obvious.

**Theorem 3.15:** \( X \) is \((1,2)^{*}-gb\)-symmetric iff \( \{ x \} \) is \((1,2)^{*}-gb\)-closed for \( x \in X \).

**Proof:** Assume that \( \{ x \} \subset \cup (1,2)^{*}-gb-cl(\{ y \}) \) but \( y \notin \cup (1,2)^{*}-gb-cl(\{ x \}) \). This implies \((1,2)^{*}-\text{gb-cl}(\{ x \})\) \( y \). Hence the set \( \{ y \} \) is a subset of \((1,2)^{*}-\text{gb-cl}(\{ x \})\). This \((1,2)^{*}-\text{gb-cl}(\{ y \})\) is a subset of \((1,2)^{*}-\text{gb-cl}(\{ x \})\). Now \((1,2)^{*}-\text{gb-cl}(\{ x \})\) contains \( x \) which is a contradiction.

Conversely, suppose \( \{ x \} \subset \cup (1,2)^{*}-gbO(X) \) but \((1,2)^{*}-\text{gb-cl}(\{ x \})\) \( y \). Hence \( \{ y \} \) is \((1,2)^{*}-\text{gb-cl}(\{ x \})\). This \((1,2)^{*}-\text{gb-cl}(\{ y \})\) is \((1,2)^{*}-\text{gb-cl}(\{ x \})\) contains \( x \) but not \( y \). Hence \( X \) is \((1,2)^{*}-gb-T_{1}\) space.

**Remark 3.17:** If \( X \) is \((1,2)^{*}-gb-T_{4}\), then \( X \) is \((1,2)^{*}-gb-T_{1}\), \( \pi \).

**Corollary 3.19:** If \( X \) is \((1,2)^{*}-gb-T_{4}\), then \( X \) is \((1,2)^{*}-gb-T_{0}\).

**Proof:** By corollary 3.18 and remark 3.17, it suffices to prove \( \Rightarrow \) (2). Let \( x \neq y \) and by \((1,2)^{*}-gb-T_{0}\), there exists \( x \in G_{1} \subset \{ y \} \). Then \( y \notin \{ y \} \). Hence \( y \notin \{ y \} \). There exists \( G_{2} \subset \{ y \} \). This \((1,2)^{*}-\text{gbO}(X)\) such that \( G_{2} \subset \{ y \} \). Hence \( X \) is \((1,2)^{*}-gb-T_{4}\) space.

**Definition 3.20:** A space \( X \) is \((1,2)^{*}-gb-T_{4}\) if for each pair of distinct points \( x \) and \( y \) in \( X \), there exists a \((1,2)^{*}-gb\)-open set \( U \) and a \((1,2)^{*}-gb\)-open set \( V \) such that \( x \in U \) and \( y \notin V \).

**Remark 3.21:** If \( X \) is \((1,2)^{*}-gb-T_{4}\) space, then \( \pi \) is \((1,2)^{*}-gb-T_{4}\).

**Definition 3.22:** Let \( X \) be a bitopological space. Let \( x \) be a point of \( X \) and \( G \) be a subset of \( X \). Then \( G \) is called an \((1,2)^{*}-gb\)-neighbourhood of \( x \) (briefly \((1,2)^{*}-gb\)-nebd of \( x \)) if there exists an \((1,2)^{*}-gb\)-open set \( U \) of \( X \) such that \( x \in U \).

**Theorem 3.23:** For a bitopological space \( X \), the following are equivalent:
1. \( X \) is \((1,2)^{*}-gb-T_{2}\).
2. If \( x \in X \), then for each \( y \neq x \), there exists \((1,2)^{*}-gb\)-nebd of \( x \) such that \( y \notin \{ x \} \).
3. If \( x \in X \), then for each \( y \neq x \), there is \((1,2)^{*}-gb\)-open set \( U \) containing \( x \) such that \( y \notin \{ x \} \).

**Proof:** (1) \( \Rightarrow \) (2): Let \( x \in X \). If \( x \notin X \), then there exists \((1,2)^{*}-gb\)-nebd \( N(x) \) such that \( y \notin \{ x \} \).

(2) \( \Rightarrow \) (3): Obvious.

(3) \( \Rightarrow \) (1): Let \( x \in X \) and \( y \neq x \). By (2), \( y \notin \{ x \} \).

**Theorem 3.24:** \( X \) is \((1,2)^{*}-gb\)-irresolute open map and \( Y \) is \((1,2)^{*}-gb-T_{2}\) then \( X \) is \((1,2)^{*}-gb-T_{2}\).

**Definition 4.1:** A subset \( A \) of a bitopological space \( X \) is called \((1,2)^{*}-D\) set if there are \( U \) and \( V \) such that \( A = U \cap V \).

**Definition 4.2:** A space \( X \) is said to be
1. \((1,2)^{*}-D_{0}\) if for any pair of distinct points \( x \) and \( y \) of \( X \), there exist a \((1,2)^{*}-D\)-set in \( X \) containing \( x \) but not \( y \) and a \((1,2)^{*}-D\)-set in \( X \) containing \( y \) but not \( x \).
(ii) $\pi$-D if for any pair of distinct points $x$ and $y$ in $X$, there exists a $(1,2)^*\pi$-D-set of $X$ containing $x$ but not $y$ and a $(1,2)^*\pi$-D-set in $X$ containing $y$ but not $x$.

(iii) $(1,2)^*\pi$-D if for any pair of distinct points $x$ and $y$ of $X$, there exists disjoint $(1,2)^*\pi$-D-sets $G$ and $H$ in $X$ containing $x$ and $y$ respectively.

**Definition 4.3:** A bitopological space $X$ is said to be $(1,2)^*\pi$-D-connected if $X$ cannot be expressed as the union of two disjoint non-empty $(1,2)^*\pi$-D-sets.

**Definition 4.4:** A bitopological space $X$ is said to be $(1,2)^*\pi$-D-compact if every cover of $X$ by $(1,2)^*\pi$-D-sets has a finite subcover.

**Definition 4.5:** A subset $A$ of a bitopological space is called $(1,2)^*\pi$-D-set if there are two open sets $U, V \in (1,2)^*\pi GBO(X)$ such that $U \subseteq X$ and $A = U - V$.

Clearly every $(1,2)^*\pi$-D-open set $U$ different from $X$ is a $(1,2)^*\pi$-D-set if $A = U$ and $V = \emptyset$.

**Example 4.6:** Let $X = \{a, b, c, d\}$ and $\tau_1 = \{\varnothing, \{a\}, \{b\}, \{a, b\}, X\}$. Then $\{c\}$ is a $(1,2)^*\pi$-D-set but not a $(1,2)^*\pi$-open set.

**Example 4.7:** A space $X$ is said to be $(1,2)^*\pi$-D if there are two $(1,2)^*\pi$-D-sets $X$ containing $x$ but not $y$ or a $(1,2)^*\pi$-D-set in $X$ containing $y$ but not $x$.

$(v)$ $(1,2)^*\pi$-D if for any pair of distinct points $x$ and $y$ in $X$ there exists a $(1,2)^*\pi$-D-set of $X$ containing $x$ but not $y$ and a $(1,2)^*\pi$-D-set in $X$ containing $y$ but not $x$.

$(vii)$ $(1,2)^*\pi$-D if for any pair of distinct points $x$ and $y$ in $X$, there exists disjoint $(1,2)^*\pi$-D-sets $G$ and $H$ in $X$ containing $x$ and $y$ respectively.

**Example 4.8:** Let $X = \{a, b, c, d\}$ and $\tau_1 = \{\varnothing, \{a\}, \{b, c\}, \{a, b, c\}, X\}$. Then $\varnothing, \{a\}, \{b, c\}$, and $\{a, b, c\}$ are $(1,2)^*\pi$-D-sets in $X$. For $U$ and $V$; since $U \subseteq \{b, c\}$ and $V \subseteq \{a, b, c\}$ then we have $S \subseteq \{a, b, c\}$ is a $(1,2)^*\pi$-D-set but not a $(1,2)^*\pi$-open set.

**Example 4.9:** A space $X$ is said to be $(1,2)^*\pi$-D if for any pair of distinct points $x$ and $y$ in $X$, there exist a $(1,2)^*\pi$-D-set in $X$ containing $x$ but not $y$ or a $(1,2)^*\pi$-D-set in $X$ containing $y$ but not $x$.

**Example 4.10:** Let $X = \{a, b, c, d\}$ and $\tau_1 = \{\varnothing, \{a\}, \{b, c\}, \{a, b, c\}, X\}$. Then $\{c\}$ is a $(1,2)^*\pi$-D-set but not a $(1,2)^*\pi$-open set.

**Example 4.11:** A subset $A$ of a bitopological space is called $(1,2)^*\pi$-D if there are two open sets $U, V \in (1,2)^*\pi GBO(X)$ such that $U \subseteq X$ and $A = U - V$.

Clearly every $(1,2)^*\pi$-D-open set $U$ different from $X$ is a $(1,2)^*\pi$-D-set if $A = U$ and $V = \emptyset$.

**Remark 4.9:** A subset $A$ of a bitopological space $X$ is called $(1,2)^*\pi$-D-set if there are two open sets $U, V \in (1,2)^*\pi GBO(X)$ such that $U \subseteq X$ and $A = U - V$.

Clearly every $(1,2)^*\pi$-D-open set $U$ different from $X$ is a $(1,2)^*\pi$-D-set if $A = U$ and $V = \emptyset$.

**Example 4.12:** Let $X = \{a, b, c, d\}$ and $\tau_1 = \{\varnothing, \{a\}, \{b, c\}, \{a, b, c\}, X\}$. Then $\{c\}$ is a $(1,2)^*\pi$-D-set.

**Example 4.13:** Let $X = \{a, b, c, d\}$ and $\tau_1 = \{\varnothing, \{a\}, \{b, c\}, \{a, b, c\}, X\}$. Then $\{c\}$ is a $(1,2)^*\pi$-D-set.

**Example 4.14:** A point $x \in X$ which has $X$ as a $(1,2)^*\pi$-D-neighborhood is called $(1,2)^*\pi$-D-neighborhood.

**Example 4.15:** Let $X = \{a, b, c\}$ and $\tau_1 = \{\varnothing, \{a\}, \{b\}, \{a, b\}, X\}$. Then $\{c\}$ is a $(1,2)^*\pi$-D-set.

**Example 4.16:** For a $(1,2)^*\pi$-D-compact bitopological space $X$, the following statements hold.

(i) $X$ is a $(1,2)^*\pi$-D-space.

(ii) $X$ is a $(1,2)^*\pi$-D-space if and only if for any pair of distinct points $x$ and $y$ in $X$, the inverse image of $y$ is a $(1,2)^*\pi$-D-space in $X$.

**Example 4.17:** For a $(1,2)^*\pi$-D-space $X$, the following statements hold.

(i) $X$ is a $(1,2)^*\pi$-D-space.

(ii) $X$ is a $(1,2)^*\pi$-D-space if and only if for any pair of distinct points $x$ and $y$ in $X$, the inverse image of $y$ is a $(1,2)^*\pi$-D-space in $X$.

**Example 4.18:** For a $(1,2)^*\pi$-D-space $X$, the following statements hold.

(i) $X$ is a $(1,2)^*\pi$-D-space.

(ii) $X$ is a $(1,2)^*\pi$-D-space if and only if for any pair of distinct points $x$ and $y$ in $X$, the inverse image of $y$ is a $(1,2)^*\pi$-D-space in $X$.
Theorem 4.19: If Y is (1,2)*-gb-D, and f: X → Y is (1,2)*-gb-irresolute and bijective, then X is (1,2)*-gb-D.
Proof: Suppose Y is (1,2)*-gb-D and f is bijective, (1,2)*-gb-irresolute. Let x, y be any pair of distinct points of X. Since f is injective and Y is (1,2)*-gb-D, there exists (1,2)*-gb-D sets G_1 and G_2 of Y containing f(x) and f(y) respectively such that f(y) ≠ G_1 and f(x) ≠ G_2. By Theorem 4.9, f^1(G_1) and f^1(G_2) are (1,2)*-gb-D sets in X containing x and y respectively. Hence X is (1,2)*-gb-D.

Theorem 4.20: A topological space X is (1,2)*-gb-D if for each pair of distinct points x, y ∈ X, there exists a (1,2)*-gb-continuous surjective function f: X → Y such that f(x) ≠ f(y).
Proof: Let x and y be any pair of distinct points in X. By hypothesis, there exists a (1,2)*-gb-continuous surjective function f of a space X onto a (1,2)*-D-space Y such that f(x) ≠ f(y). Hence there exists disjoint (1,2)*-D sets S_x and S_y in Y such that f(S_x) ⊈ f(S_y) and f(f(S_x)) and f(f(S_y)) are disjoint (1,2)*-gb-D sets in X containing x and y respectively. Hence X is a (1,2)*-gb-D set.

Theorem 4.21: X is (1,2)*-gb-D iff for each pair of distinct points x, y ∈ X, there exists a (1,2)*-gb-irresolute surjective function f: X → Y, where Y is (1,2)*-gb-D space such that f(x) and f(y) are distinct.
Proof: Necessity: For every pair of distinct points x, y ∈ X, it suffices to take the identity function on X.
Sufficiency: Let x ≠ y ∈ X. By hypothesis, there exists a (1,2)*-gb-irresolute, surjective function from X onto a (1,2)*-gb-D set such that f(x) ≠ f(y). Hence there exists disjoint (1,2)*-gb-D sets G_x and G_y in Y such that f(G_x) ⊈ f(G_y) and f(G_x) and f(G_y) are disjoint (1,2)*-gb-D sets in X containing x and y respectively. Therefore X is (1,2)*-gb-D space.

Definition 4.22: A topological space is said to be (1,2)*-gb-D-connected if X cannot be expressed as the union of two disjoint non-empty (1,2)*-gb-D sets.

Theorem 4.23: If X → Y is (1,2)*-gb-continuous surjection and X is (1,2)*-gb-D-connected, then Y is (1,2)*-D-connected.
Proof: Suppose Y is not (1,2)*-D-connected. Let Y = A ∪ B where A and B are two disjoint non-empty (1,2)*-D sets in Y. Since f is (1,2)*-gb-continuous and onto, X = f^1(A) ∪ f^1(B) where f^1(A) and f^1(B) are disjoint non-empty (1,2)*-gb-D sets in X. This contradicts the fact that X is (1,2)*-gb-D-connected. Hence Y is (1,2)*-D-connected.

Theorem 4.24: If X → Y is (1,2)*-gb-irresolute surjection and X is (1,2)*-gb-D-connected, then Y is (1,2)*-gb-D-connected.
Proof: Suppose Y is not (1,2)*-gb-D-connected. Let Y = A ∪ B where A and B are disjoint non-empty (1,2)*-gb-D sets in Y. Since f is (1,2)*-gb-irresolute and onto, X = f^1(A) ∪ f^1(B) where f^1(A) and f^1(B) are disjoint non-empty (1,2)*-gb-D sets in X. This contradicts the fact that X is (1,2)*-gb-D-connected. Hence Y is (1,2)*-gb-D-connected.

Definition 4.25: A topological space is said to be (1,2)*-gb-D-connected if every cover of X by (1,2)*-gb-D sets has a finite subcover.

Theorem 4.26: If a function f: (X, τ) → (Y, ν) is (1,2)*-gb-continuous surjection and (X, τ) is (1,2)*-gb-D-compact then Y is (1,2)*-gb-D-compact.
\((1,2)^*\)-\(\text{ngb}(y)\), we have \(\{y\}\cap(1,2)^*\)-\(\text{ngb- cl}(z)\) = \(\Phi\). Since \(x \in (1,2)^*\)-\(\text{ngb-cl}\{z\}\), \((1,2)^*\)-\(\text{ngb-cl}\{x\}\) is \((1,2)^*\)-\(\text{ngb-cl}\{z\}\) and \(y \in (1,2)^*\)-\(\text{ngb-cl}(z)\) = \(\Phi\). Therefore, \((1,2)^*\)-\(\text{ngb-cl}(x)\) \(\neq (1,2)^*\)-\(\text{ngb-cl}(y)\). Now Ker \((1,2)^*\)-\(\text{ngb}(x)\) \(\neq\) Ker \((1,2)^*\)-\(\text{ngb}(y)\) implies that \((1,2)^*\)-\(\text{ngb-cl}(x)\) \(\neq (1,2)^*\)-\(\text{ngb-cl}(y)\).

\((2) \Rightarrow (1):\) Suppose that \((1,2)^*\)-\(\text{ngb-cl}(x)\) \(\neq (1,2)^*\)-\(\text{ngb-cl}(y)\).

Then there exists a point \(z \in X\) such that \(z \in (1,2)^*\)-\(\text{ngb-cl}(x)\) and \(z \notin (1,2)^*\)-\(\text{ngb-cl}(y)\). Then, there exists an \((1,2)^*\)-\(\text{ngb-open}\) set containing \(z\) and hence containing \(x\) but not \(y\), i.e., \(y \notin\) Ker \((1,2)^*\)-\(\text{ngb}(y)\). Hence Ker \((1,2)^*\)-\(\text{ngb}(y)\) \(\neq\) Ker \((1,2)^*\)-\(\text{ngb}(y)\).

\textbf{Definition 5.7:} A bitopological space is said to be \((1,2)^*\)-\(\text{ngb-R}_R\) if \((1,2)^*\)-\(\text{ngb-cl}\{x\}\) \(\subseteq G\) whenever \(x \in G \cap (1,2)^*\)-\(\text{ngbBO}(X)\).

\textbf{Definition 5.8:} A bitopological space is said to be \((1,2)^*\)-\(\text{ngb-R}_R\) if for any \(x, y \in X\) with \((1,2)^*\)-\(\text{ngb-cl}(x)\) \(\neq (1,2)^*\)-\(\text{ngb-cl}(y)\), there exists disjoint \((1,2)^*\)-\(\text{ngb-open}\) sets \(U\) and \(V\) such that \((1,2)^*\)-\(\text{ngb-cl}(x)\) \(\subseteq U\) and \((1,2)^*\)-\(\text{ngb-cl}(y)\) \(\subseteq V\).

\textbf{Definition 5.9:} A bitopological space \((1,2)^*\)-\(\text{R}_R\) is \((1,2)^*\)-weakly \(\pi\)-\(\text{ngb-R}_R\) iff \((1,2)^*\)-\(\text{ngb-cl}\{x\}\) \(\subseteq G\) whenever \(x \in G \cap (1,2)^*\)-\(\text{ngbBO}(X)\).

\textbf{Example 5.10:} Let \(X=\{a,b,c,d\}\). \(\tau=\{(\Phi,\{b\},\{a,b\},\{b,c\},\{b,d\},X\), \(\tau_1=\{\Phi,\{a,b\},X\}\). \((1,2)^*\)-\(\text{ngbBO}(\tau)=\{P(X),X\}\). Then \(X\) is \((1,2)^*\)-\(\text{ngb-R}_R\) if and only if \((1,2)^*\)-\(\text{ngb-R}_R\).

\textbf{Remark 5.11:} Every \((1,2)^*\)-\(\text{ngb-R}_R\) space is \((1,2)^*\)-\(\text{ngb-R}_0\) space.

Let \(U\) be a \((1,2)^*\)-\(\text{ngb-open}\) set such that \(x \in U\). Then since \(x \in (1,2)^*\)-\(\text{ngb-cl}\{y\}\), \((1,2)^*\)-\(\text{ngb-cl}(x)\) \(\neq (1,2)^*\)-\(\text{ngb-cl}(y)\). Hence there exists an \((1,2)^*\)-\(\text{ngb-open}\) set \(V\) such that \(y \in V\) such that \((1,2)^*\)-\(\text{ngb-cl}(y)\) \(\subseteq V\) and if \(x \neq y\) \(\Rightarrow\) \(y \notin (1,2)^*\)-\(\text{ngb-cl}(x)\). Hence \((1,2)^*\)-\(\text{ngb-cl}(x)\) \(\subseteq U\). Hence \(x\) is \((1,2)^*\)-\(\text{ngb-R}_R\).

\textbf{Theorem 5.12 :} \((1,2)^*\)-\(\text{ngb-R}_R\) iff given \(x \neq y\); \((1,2)^*\)-\(\text{ngb-cl}(x)\) \(\neq (1,2)^*\)-\(\text{ngb-cl}(y)\).

Proof: Let \(X\) be \((1,2)^*\)-\(\text{ngb-R}_R\) and let \(x \neq y\) \(\in X\). Suppose \(U\) is \((1,2)^*\)-\(\text{ngb-open}\) set containing \(x\) but not \(y\), then \(y \in (1,2)^*\)-\(\text{ngb-cl}(y)\) \(\subseteq X\)-\(U\) and hence \(x \notin (1,2)^*\)-\(\text{ngb-cl}(y)\). Hence \((1,2)^*\)-\(\text{ngb-cl}(x)\) \(\neq (1,2)^*\)-\(\text{ngb-cl}(y)\).

Conversely, let \(x \neq y\) \(\in X\) such that \((1,2)^*\)-\(\text{ngb-cl}(x)\) \(\neq (1,2)^*\)-\(\text{ngb-cl}(y)\). This implies \((1,2)^*\)-\(\text{ngb-cl}(x)\) \(\subseteq (1,2)^*\)-\(\text{ngb-cl}(y)\). This is true for every \((1,2)^*\)-\(\text{ngb-cl}(x)\). Then \((1,2)^*\)-\(\text{ngb-cl}(x)\) \(\subseteq U\) where \(x \in (1,2)^*\)-\(\text{ngb-cl}(x)\) \(\subseteq \bigcup(1,2)^*\)-\(\text{ngbBO}(X)\). This implies \((1,2)^*\)-\(\text{ngb-cl}(x)\) \(\subseteq U\) where \(x \in \bigcup(1,2)^*\)-\(\text{ngbBO}(X)\). Hence \(X\) is \((1,2)^*\)-\(\text{ngb-R}_R\).
We prove the result using theorem 5.13. Let \( x \in (1,2)^*\text{-}\text{ngb-cl}(\{y\}) \) and by theorem 5.14, \( y \in \text{Ker}(1,2)^*\text{-}\text{ngb}(x) \). Since \( x \in (1,2)^*\text{-}\text{ngb-cl}(\{x\}) \) and \( (1,2)^*\text{-}\text{ngb-cl}(\{x\}) \) is \( (1,2)^*\text{-}\text{ngb-closed} \), then by (iv) we get \( y \in \text{Ker}(1,2)^*\text{-}\text{ngb}(x) \). Conversely, let \( y \in (1,2)^*\text{-}\text{ngb-cl}(\{x\}) \). By lemma 5.5, \( x \in \text{Ker}(1,2)^*\text{-}\text{ngb}(\{y\}) \). Since \( y \in (1,2)^*\text{-}\text{ngb-cl}(\{y\}) \) and \( (1,2)^*\text{-}\text{ngb-cl}(\{y\}) \) is \( (1,2)^*\text{-}\text{ngb-closed} \), then by (iv) we get \( x \in \text{Ker}(1,2)^*\text{-}\text{ngb}(\{y\}) \). Thus \( y \in (1,2)^*\text{-}\text{ngb-cl}(\{y\}) \). By theorem 5.14, we prove that \( x \) is \( (1,2)^*\text{-}\text{ngb-R}_0 \) space.

**Theorem 5.19:*** A bitopological space \( X \) is \( (1,2)^*\text{-}\text{ngb-R}_1 \) iff for \( x,y \in X \), Ker \((1,2)^*\text{-}\text{ngb}(x)\nneq(1,2)^*\text{-}\text{ngb}(y)\), there exists disjoint \((1,2)^*\text{-}\text{ngb-open set}\) U and V such that \((1,2)^*\text{-}\text{ngb-cl}(\{x\})\subseteq U \) and \((1,2)^*\text{-}\text{ngb-cl}(\{y\})\subseteq V \).

**Proof:** It follows from lemma 5.5.

**Theorem 5.20:*** A bitopological space \( X \) is \( (1,2)^*\text{-}\text{ngb-T}_3 \) if and only if it is \((1,2)^*\text{-}\text{ngb-T}_1 \) and \((1,2)^*\text{-}\text{ngb-R}_1 \).

**Proof:** If \( X \) is \( (1,2)^*\text{-}\text{ngb-T}_3 \), then it is \( (1,2)^*\text{-}\text{ngb-T}_1 \). If \( x,y \in X \) such that \((1,2)^*\text{-}\text{ngb-cl}(\{x\})\nneq(1,2)^*\text{-}\text{ngb-cl}(\{y\})\), then \( x \neq y \). Hence there exists disjoint \((1,2)^*\text{-}\text{ngb-open set}\) U and V such that \(x \in U \) and \(y \in V \). This implies \((1,2)^*\text{-}\text{ngb-cl}(\{x\})\subseteq U \) and \((1,2)^*\text{-}\text{ngb-cl}(\{y\})\subseteq V \). Hence \( X \) is \((1,2)^*\text{-}\text{ngb-R}_1 \).

**Theorem 5.21:*** A bitopological space \( X \) is said to be weakly \( (1,2)^*\text{-}\text{ngb-R}_0 \) if \( (1,2)^*\text{-}\text{Ker}_{\text{ngb}}(x) \neq X \) for every \( x \in X \).

**Proof:** Suppose that the space \( X \) is weakly \((1,2)^*\text{-}\text{ngb-R}_0 \). Assume that there is a point \( y \) in \( X \) such that \((1,2)^*\text{-}\text{Ker}_{\text{ngb}}(y) \neq X \). Then \( y \in O \) where \( O \) is some proper \((1,2)^*\text{-}\text{ngb-open set} \) of \( X \). This implies \( y \in \bigcap \{x \mid (1,2)^*\text{-}\text{ngb-cl}(\{x\}) \} \) which is a contradiction.

Conversely, Assume \( (1,2)^*\text{-}\text{Ker}_{\text{ngb}}(x) \neq X \) for every \( x \in X \). If there is a point \( y \in X \) such that \( y \in \bigcap \{x \mid (1,2)^*\text{-}\text{ngb-cl}(\{x\}) \} \), then every \((1,2)^*\text{-}\text{ngb-open set} \) containing \( y \) must contain every point of \( X \). This implies the unique \((1,2)^*\text{-}\text{ngb-open set} \) containing \( y \). Hence \((1,2)^*\text{-}\text{Ker}_{\text{ngb}}(y) = X \), which is a contradiction. Thus \( X \) is weakly \((1,2)^*\text{-}\text{ngb-R}_0 \).

**Example 5.22:** Let \( X = \{a,b,c,d\}, \{a,b\} \cap \{a,c\} = \{a\}, \text{and} \{a\} \subset \emptyset \). Then \( \text{Ker}_{\text{ngb}}(X) = \{a\} \). Thus \( \text{Ker}_{\text{ngb}}(X) \neq X \).

**Conclusion:** A study on new separation axioms called ngb-separation axioms using the \((1,2)^*\text{-}\text{ngb-open set} \) in bitopological spaces has been done. Also some results of \((1,2)^*\text{-}\text{ngb-T}_3 \), \((1,2)^*\text{-}\text{ngb-D}_0 \), where \( i = 0,1,2 \), and \((1,2)^*\text{-}\text{ngb-R}_0 \), are studied in this paper.
References