**Abstract**

\( P_{4k+1} \)-factorization of a complete bipartite graph for an integer was studied by Wang [1]. Further, Beiliang [2] extended the work of Wang [1], and studied the \( P_{2k} \)-factorization of complete bipartite multigraphs. For even value of \( k \) in \( P_k \)-factorization, the spectrum problem is completely solved [1, 2, 3]. However for odd value of \( k \) i.e. \( P_3, P_5, P_7 \) and \( P_9 \), the path factorization have been studied by a number of researchers [4, 5, 6, 7]. Again, \( P_3 \)-factorizations of complete bipartite multigraphs and symmetric complete bipartite multi-digraphs were studied by Wang and Beiliang [8]. Also, Beiliang and Wang have shown that Ushio conjecture is true for complete bipartite graphs [9]. In the present paper we shall show that Ushio conjecture is also true for \( 4k + 1 \) factorization of complete bipartite graphs. That is, we shall prove that a necessary and sufficient condition for the existence of a \( P_{4k+1} \)-factorization of \( K_{m,n} \) is

\[
\begin{align*}
(1) \quad & (2k+1)m \geq 2kn, \\
(2) \quad & (2k+1)n \geq 2km, \\
(3) \quad & m + n \equiv 0 \mod 4k + 1, \\
(4) \quad & [(2k+1)(m+n)]/(4k(m+n)) \text{ is an integer.}
\end{align*}
\]

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**Introduction**

Ushio conjecture [11] for path factorization of complete bipartite graphs is as follows:

If \( k \) is odd, and \( m \) and \( n \) be positive integers, then \( K_{m,n} \) has \( P_k \)-factorization if and only if:

\[
\begin{align*}
(1) \quad & (k-1)m \geq kn, \\
(2) \quad & (k-1)n \geq km, \\
(3) \quad & m + n \equiv 0 \mod k, \\
(4) \quad & mn/(k-1)(m+n) \text{ is an integer.}
\end{align*}
\]

In this paper, we shall prove that Ushio conjecture is true for the path factorization of \( P_{4k+1} \)-factorization of complete bipartite graphs, that is we shall prove the theorem given below.

**Theorem 1:** Let \( k, m, n \) be positive integers, there exist a \( P_{4k+1} \)-factorization of \( K_{m,n} \) if and only if:

\[
\begin{align*}
(1) \quad & (2k+1)m \geq 2kn, \\
(2) \quad & (2k+1)n \geq 2km, \\
(3) \quad & m + n \equiv 0 \mod 4k + 1, \\
(4) \quad & [(4k+1)(m+n)]/(4k(m+n)) \text{ is an integer.}
\end{align*}
\]

**Mathematical Analysis**

We first give the proof of necessity of theorem 1, which is given in theorem 2. The sufficiency of theorem 1 is proved by theorem 3.

**Theorem 2:** Let \( k, m, n \) be positive integers. Then for \( P_{4k+1} \)-factorization it is necessary that:

\[
\begin{align*}
(1) \quad & (2k+1)m \geq 2kn, \\
(2) \quad & (2k+1)n \geq 2km, \\
(3) \quad & m + n \equiv 0 \mod 4k + 1, \\
(4) \quad & [(4k+1)(m+n)]/(4k(m+n)) \text{ is an integer.}
\end{align*}
\]

**Proof:** Let \( r \) be the number of \( P_{4k+1} \)-factor in the factorization and \( \theta \) be the number of copies of \( P_{4k+1} \) in any factor.

Then \( \theta = \frac{m+n}{4k+1} \), and \( \gamma = \frac{(4k+1)m}{4k(m+n)} \)
Let \(a\) and \(b\) be the number of copies of \(P_{4k+1}\) with its end points in \(Y\) and \(X\) in a particular \(P_{4k+1}\)-factor respectively.

Then,

\[
(2k)a + (2k + 1)b = m, \text{ and } (2k + 1)a + (2k)b = n.
\]

Hence,

\[
a = \frac{(2k+1)m-(2k)n}{4k+1}, \quad \text{and} \quad b = \frac{(2k+1)m-(2k)n}{4k+1}.
\]

Conditions (1) and (2) are therefore, necessary. This proves the necessity of the theorem 1.

Now we will prove the sufficiency of theorem 1. Which is given by theorem 3.

**Theorem 3:** Let \(k, m, n\) be positive integers. Then for \(P_{4k+1}\)-factorization, it is sufficient that:

1. \((2k + 1)m \geq 2kn\),
2. \((2k + 1)n \geq 2km\),
3. \(m + n \equiv 0 \pmod{4k + 1}\),
4. \((4k + 1)mn/[4k(m + n)]\) is an integer.

The proof of this theorem, consist of the following lemmas.

**Lemma 1:** Let \(a, b, p\) and \(q\) be positive integers. If \(\gcd(ap, bq) = 1\), then \(\gcd(ab, ap + bq) = 1\).

We prove the following result which is used later in the paper.

**Lemma 2:** If \(K_{m,n}\) has a \(P_{4k+1}\)-factorization, then \(K_{sm,sn}\) has a \(P_{4k+1}\)-factorization for every positive integer \(s\).

**Proof:** Let \(F_1, F_2, \ldots, F_s\) be a \(1\)-factorable \([10]\) and \(\{F_1, F_2, \ldots, F_s\}\)

be a \(1\)-factorization of it. For each \(i\) with \(1 \leq i \leq s\), replace every edge of \(F_i\) by a \(K_{m,n}\) to get a spanning subgraph \(G_i\) of \(K_{sm,sn}\) such that the graph \(G_i\)'s \(\{1 \leq i \leq s\}\) are pair wise edge disjoint and there union is \(K_{sm,sn}\). Since \(K_{m,n}\) has a \(P_{4k+1}\)-factorization, it is clear that the \(G_i\) is also \(P_{4k+1}\)-factorable and hence \(K_{sm,sn}\) is also \(P_{4k+1}\)-factorable.

Lemma 2 implies that there are three cases to consider.

Case (1) \(2km = (2k + 1)n\)

In this case, let

\[
F_j = \{x_{i+j}y_i, x_{i+j}y_{i+j}; 1 \leq i \leq 4k, 1 \leq j \leq 2(k + 1)\}.
\]

It is easy to see that it is a \(P_{4k+1}\)-factor of \(K_{4k+3}\).

Then \(U_{1s}F_j\) is a \(P_{4k+1}\)-factor of \(K_{4k+2}, K_{m,n}\) has a \(P_{4k+1}\)-factorization.

Case (2) \((2k + 1)m = 2kn\)

Obviously, \(K_{m,n}\) has a \(P_{4k+1}\)-factorization.

Case (3) \((2k + 1)m > (2k)n\)

In this case, let

\[
a = \frac{(2k+1)n-2km}{4k+1}, \quad b = \frac{(2k+1)m-2kn}{4k+1},
\]

Conditions (1) and (2) are necessary. This proves the necessity of the theorem 1.

We have

\[
\gcd((2k)a, (2k + 1)b) = d.
\]

Then, \(2ka = dp\) and \((2k + 1)b = dq\), where \(\gcd(p, q) = 1\).

Therefore,

\[
\gamma = \frac{dpq}{2k((2k + 1)p + 2kq)}.\]

These equalities imply the following equalities:
Now we can establish the following lemma.

Lemma 3:
If
\[\gcd(p, q) = 1, \quad \gcd(p, 2k + 1) = 1,\]

where
\[1 \leq i \leq \mu, \quad 0 < j_i < \nu, \]

then
\[\frac{p}{2\nu^2}q^2 \quad \text{and} \quad s' = u^2v^2w^2q'.\]

Case (1): If \(t' \equiv 1 \pmod{2}\) and \(v'w' \equiv 1 \pmod{2}\), then for some positive integer \(z'\):

\[m = 2\text{stut}(suw^2wp + s'u'v^2w'^2, svw^2q),\]

\[n = 2\text{stut}(t'vwp + t'v'w'q),\]

\[\alpha = \text{stut}(svw^2p + st^2uq),\]

\[\beta = \text{stut}(t'vwp + t'v'w'q),\]

\[r = t'w'(suw^2wp + s'u'v^2w'^2q),\]

\[\frac{c}{d} = \frac{2\text{stut}(t'vwp + t'v'w'q)}{v'w'}.\]

Proof. We are now giving the proof of each case of lemma 3.

\[\gcd(p, q) = 1, \quad \gcd(p, 2k + 1) = 1.\]

Case (3): If \(t' \equiv 1 \pmod{2}\) and \(v'w' \equiv 0 \pmod{2}\), then for some positive integer \(z'\):

\[m = 2\text{stut}(suw^2wp + s'u'v^2w'^2, tv'wp + t'v'w'q),\]

\[n = suwv(w's't^2u'p + st^2uq),\]

\[\alpha = \text{stut}(tv'wp + t'v'w'q),\]

\[\beta = \text{stut}(tv'wp + t'v'w'q),\]

\[r = t'w'(suw^2wp + s'u'v^2w'^2q),\]

\[\frac{c}{d} = \frac{2\text{stut}(tv'wp + t'v'w'q)}{v'w'}.\]

Also, let
\[\frac{p}{2\nu^2}q^2 = \text{stut}(suw^2wp + s'u'v^2w'^2q).\]

Case (2): If \(t' \equiv 0 \pmod{2}\) and \(v'w' \equiv 1 \pmod{2}\), then for some positive integer \(z'\):

\[m = 2\text{stut}(suw^2wp + s'u'v^2w'^2, tv'wp + t'v'w'q),\]

\[n = suwv(w's't^2u'p + st^2uq),\]

\[\alpha = \text{stut}(tv'wp + t'v'w'q),\]

\[\beta = \text{stut}(tv'wp + t'v'w'q),\]

\[r = t'w'(suw^2wp + s'u'v^2w'^2q),\]

\[\frac{c}{d} = \frac{2\text{stut}(tv'wp + t'v'w'q)}{v'w'}.\]

Depending on the values of parameters \(t'\) and \(v'w'\) the proof of three cases of lemma 3 are as follows.

Case (1): If \(t' \equiv 1 \pmod{2}\) and \(v'w' \equiv 1 \pmod{2}\), then for some positive integer \(z'\):

\[m = 2\text{stut}(suw^2wp + s'u'v^2w'^2, tv'wp + t'v'w'q),\]

\[n = suwv(w's't^2u'p + st^2uq),\]

\[\alpha = \text{stut}(tv'wp + t'v'w'q),\]

\[\beta = \text{stut}(tv'wp + t'v'w'q),\]

\[r = t'w'(suw^2wp + s'u'v^2w'^2q),\]

\[\frac{c}{d} = \frac{2\text{stut}(tv'wp + t'v'w'q)}{v'w'}.\]

Case (2): If \(t' \equiv 0 \pmod{2}\) and \(v'w' \equiv 1 \pmod{2}\), then for some positive integer \(z'\):

\[m = 2\text{stut}(suw^2wp + s'u'v^2w'^2, tv'wp + t'v'w'q),\]

\[n = suwv(w's't^2u'p + st^2uq),\]

\[\alpha = \text{stut}(tv'wp + t'v'w'q),\]

\[\beta = \text{stut}(tv'wp + t'v'w'q),\]

\[r = t'w'(suw^2wp + s'u'v^2w'^2q),\]

\[\frac{c}{d} = \frac{2\text{stut}(tv'wp + t'v'w'q)}{v'w'}.\]

Case (3): If \(t' \equiv 1 \pmod{2}\) and \(v'w' \equiv 0 \pmod{2}\), then for some positive integer \(z'\):

\[m = 2\text{stut}(suw^2wp + s'u'v^2w'^2, tv'wp + t'v'w'q),\]

\[n = suwv(w's't^2u'p + st^2uq),\]

\[\alpha = \text{stut}(tv'wp + t'v'w'q),\]

\[\beta = \text{stut}(tv'wp + t'v'w'q),\]

\[r = t'w'(suw^2wp + s'u'v^2w'^2q),\]

\[\frac{c}{d} = \frac{2\text{stut}(tv'wp + t'v'w'q)}{v'w'}.\]
Since \( \gcd(2, v'w') = \gcd(stuv, v'w') = 1 \) and
\( \gcd(vwtp' + tw'q', v'w') = 1 \).

hence \( \frac{\frac{s}{(v'w')}}{t'} \) is an integer. Let \( z_2 = \frac{\frac{s}{(v'w')}}{t'} \), then
\[
\frac{2\text{suww}w'(s' t^2 u' p + st^2 u' q)(t'suwp + ts'v'w'q)z_2}{t'} = z_2.
\]

Since,
\[
\gcd(2, t') = 2, \gcd(suww, t') = \gcd(t'suwp + ts'v'w'q, t') = 1
\]
also
\[
\gcd(s't^2 u' p + st^2 u' q', t') = 1, \text{ therefore } \frac{\frac{s}{t'}}{z_2}
\]
is an integer.

Let \( z_2 = \frac{\frac{s}{t'}}{z_2} \), then the equalities in (2) hold.

Case (3): \( t' \equiv 1 \pmod{2} \) and \( v'w' \equiv 0 \pmod{2} \)
\[
\gcd(2, v'w') = 2, \gcd(stuv, v'w') = 1
\]

hence \( \frac{\frac{s}{(v'w')}}{t'} \) is an integer.

Let \( z_2 = \frac{\frac{s}{(v'w')}}{t'} \), then
\[
\frac{\text{suww}w'(s't^2 u' p + st^2 u' q)(t'suwp + ts'v'w'q)z_2}{t'} = z_2.
\]

Since,
\[
\gcd(2, t') = \gcd(suww, t') = \gcd(t'suwp + ts'v'w'q, t') = 1
\]
and
\[
\gcd(s't^2 u' p + st^2 u' q', t') = \gcd(s't^2 u' p + st^2 u' q', t') = 1
\],

therefore \( \frac{\frac{s}{t'}}{z_2} \) is an integer.

Let \( z_2 = \frac{\frac{s}{t'}}{z_2} \), then the equalities in case (3) hold.

This proves the lemma 3.

We now give the direct construction of case (1) by taking
\( z_1 = 1 \) in lemma 3. We will call this as lemma 4.

Lemma 4. For any positive integers \( s, t, u, v, w, s', t', u', v', w', p \) and \( q, \)
let
\[
m = 4stuwp'(suv^2wp + s'u've'w'q)(vwt'p + tw'q),
\]
\[
n = 2suwww'(s't^2wp + st^2uq)(vp't's + qts'v'w')q.
\]

Then \( K_{mn} \) has a \( P_{4k+1} \)-factorization.

Proof. Let \( a = 2suwwt'(t'vwp + tw'q), b = 2suwwv'(qwt'p + tw'q) \)
and \( r = t'vwp'(suv^2wp + s'u've'w'q)(vwt'+tw'q), \)
then
\[
r_1 = t'(suv^2wp + s'u've'w'q), \quad \text{and} \quad r_2 = v'w'(s't^2u'p + st^2uq).
\]

Let \( X \) and \( Y \) be two partite set of \( K_{mn} \) such that
\[
X = \{x_{i,j}; 1 \leq i \leq r_1, 1 \leq j \leq m_1\}
\]
and
\[
Y = \{y_{i,j}; 1 \leq i \leq r_2, 1 \leq j \leq n_0\}.
\]

where \( m_1 = \frac{m}{r_2} = 4stuwp't + tw'q'w' \)
and
\( n_0 = \frac{n}{r_2} = 2suww(vwp t's + qts'v'w'). \)

Now for each \( i, x, y, z \) and \( z' \),
\[
1 \leq i \leq t'p, 1 \leq x \leq vw, 1 \leq y \leq suwv, 1 \leq z \leq t \text{ and } 0 \leq x' \leq 1,
\]
let
\[
f(i, x, y) = suv^2w^2(i - 1) + suw(x - 1) + y,
\]
\[
g(i, y, z, x') = st'v'w'(i - 1) + suw(x - 1) + y + x' \text{ and}
\]
\[
b(i, x, y, x') = 2suw(i - 1) + 2suw(vwt'p + tw'q')(x - 1) + 2y + x' - 1,
\]
here \( suwwv = 1 = s't'v'u'w' \) and set
\[
E_i = \{(x_{i,j}, y, z, x'); j \in \{i + 1, j + 1, k + 1\}\} \quad 1 \leq j \leq 4suw(vwt'p + tw'q')(x - 1) + 2y + x' - 1,
\]
for each \( i, x, y, z, x' \),
\[
1 \leq i \leq t'v'q', 1 \leq x \leq stv, 1 \leq y \leq vw, 1 \leq z \leq t \text{ and } 0 \leq x' \leq 1,
\]
Let
\[
\phi(i, x, y) = suv^2w^2(i - 1) + suw(x - 1) + y + x,
\]
\[
\psi(i, x, z) = st'v'u'w'(i - 1) + suw(x - 1) + x' \text{ and}
\]
\[
\theta(i, x, y, x') = 2suw(i - 1) + 2z + 4suw(vwt'p + tw'q')(x - 1) + x + x' + 1,
\]
and let
\[
E_i' = \{(x_{i,j}, y, z, x'); j \in \{i + 1, j + 1, k + 1\}\} \quad 1 \leq j \leq 4suw(vwt'p + tw'q')(x - 1) + 2y + x' - 1,
\]
for each \( i, x, y, z, x' \),
\[
1 \leq i \leq 4suw(vwt'p + tw'q')(y - 1) + x' - 1,
\]
and let
\[
F = \bigcup_{1 \leq i \leq t'} E_i, \text{ then it is easy to see that the graph } F
\]
is a \( P_{4k+1} \)-factor of \( K_{mn} \). Define a bijection \( \sigma \)
from \( X \cup Y \) onto \( X \cup Y \) in such a way that
\[
\sigma(x_{i,j}) = x_{i+1,j}, \sigma(y_{i,j}) = y_{i+1,j}.
\]

For each \( i \in \{1, 2, ..., r_2\} \) and each \( j \in \{1, 2, ..., r_2\} \),
let
\[
F_{i,j} = \{\sigma(x) \sigma(y): x \in X, y \in Y, xy \in F\}. \]
It is easy to show that the graphs
\[ F_{k,l}(1 \leq i \leq r_1, 1 \leq j \leq r_2) -factor of K_{m,n} \]
and there is a
\[ P_{4k+1} -factorization of K_{m,n}. \]

Thus \((F_{k,l}: 1 \leq i \leq r_1, 1 \leq j \leq r_2) -factorization of K_{m,n}. \)

This proves the lemma 4. Similarly we can prove the direct
constructions of cases (2) and (3).

Proof (Theorem 3):
Applying lemmas 2-4, we see that for the parameters \( k, m \) and \( n \) satisfying conditions (1)-(4) in theorem 1, \( K_{m,n} \) has a
\[ P_{4k+1} -factorization. \]
This proves the sufficiency of the
conditions given in theorem 3.

Proof (Theorem 1):
Combining theorem 2 and 3, we complete the proof of theorem
1. This proves that Ushio conjecture for
\[ P_{4k+1} -factorization of K_{m,n} \]
is true.

References: