Bi-Edge – graceful and global edge – graceful labeling of some graphs

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ABSTRACT
Lo[14] introduced the notion of edge-graceful graphs. In this paper, we extend this to bi-edge and global edge-graceful graphs. Here, we completely characterized the global edge-graceful graphs.

Keywords
Edge-graceful,
Bi-edge-graceful and global edge-graceful labeling
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Introduction
Graphs have many applications in various areas of Computer science. Parallel computers can be modeled by a graph, in which the vertices of the graph represent the processors and the edges represent the communications among processors.

The main objective is to derive a graph from the set of objects and to order(rank) the nodes according to their relations with others in the graph. The basic idea of ranking approach is to share the total processes among p nodes and to assign the tasks according to their modulo ranking by finding the strength of the node as the sum of the weights of the edges with which it is incident.

The mathematical idea for performing this task in the network topology is to check whether the topology is edge-graceful or not. Here, the weights of the edges represent the labels of the edges.

A graph labeling is an assignment of integers to the vertices or edges or both subject to certain conditions. If the domain of the mapping is the set of vertices or edges then the labeling is called vertex or edge labeling. Graph labelings were first introduced in the late 1960’s. In the recent years, dozens of graph labeling techniques have been studied in over 1000 papers.

Labeled graphs serve as useful models for a broad range of applications such as coding theory, x-ray crystallography, radar, astronomy, circuit design, communication network addressing, data base management, and models for constraint programming over finite domains [1, 2].

By a (p, q) graph G, we mean a graph G = (V, E) with |V| = p and |E| = q. Most graph labeling methods trace their origin from the definition introduced by Rosa in 1967. Golomb [8] subsequently called such labeling as graceful.

In 1985, Lo [14] introduced the edge - graceful graph which is a dual notion of graceful labeling.

A (p,q) graph G is said to have an edge-graceful labeling if there exists an injection f from the edge set to {1,2,3, ...,q} so that the induced mapping $f^e$ defined on the vertex set defined by $f^e(x) = \Sigma[f(xy): xy is an edge](mod p)$ are distinct.

A graph G is said to be edge - graceful if it admits an edge graceful labeling. Lo[14], proved that G is edge - graceful if
\[ q(q+1) \equiv \frac{(p+1)p}{2} (mod p). \]

In this paper, we introduce the bi-edge graceful and global edge-graceful labeling of graphs which are extension of edge-graceful labeling. Here we investigated these types of graphs and we have completely characterized the global edge-graceful graphs.

For global edge – gracefulfulness of a (p,q) graph G, we assume that both G and $G^c$ are connected. In other words, q takes the value between $p-1$ and $(p-1)(p-2)/2$. Throughout this paper, we assume that G is a connected simple undirected graph.

Bi-edge-graceful Definition
A graph G is said to be bi-edge - graceful if both G and its line graph $L(G)$ are edge - graceful.

Theorem
The path $P_n$ is not bi-edge - graceful.

Proof
Let $P_n$ be the path of n vertices and n-1 edges.
Case (1): n is odd
Let $L(P_n) = P_{n-1}$ which is an even order tree and is not edge – graceful by the necessary conditions of Lo and hence $P_n$ is not bi-edge – graceful.
Case (2): n is even
In this case, $P_n$ is of even order and again by the same arguments it is not bi-edge – graceful.

Theorem
The graph $C_{2n+1}$ is bi-edge - graceful.

Proof
Let $C_{2n+1}$ be the cycle with $2n + 1$ vertices and $2n + 1$ edges.

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Case (1): $C_{2n+1}$ is edge - graceful.
Let $v_1, v_2, v_3, \ldots, v_{2n+1}$ be the vertices of $C_{2n+1}$, and the edges $e_i$ are defined as follows (see fig.2.1).

**Fig. 2.1** $C_{2n+1}$ with ordinary labeling

$e_i = (v_i, v_{i+1})$ for $1 \leq i \leq 2n$
$e_{2n+1} = (v_{2n+1}, v_1)$

First we label the edges of $C_{2n+1}$

$f(e_i) = i$ for $1 \leq i \leq 2n + 1$

Then the induced vertex labels are

$$f^+(v_i) = 2i - 1 \quad \text{for} \quad 1 \leq i \leq n$$

$$f^+(v_i) = 2(i - n - 1) \quad \text{for} \quad n + 1 \leq i \leq 2n + 1$$

These labels are arranged in order. Let

$$A = \{f^+(v_i) \mid 1 \leq i \leq n\} = \{1, 3, 5, \ldots, 2n - 1\}$$

$$B = \{f^+(v_i) \mid n + 1 \leq i \leq 2n + 1\} = \{0, 2, 4, \ldots, 2n\}$$

$$f^+(V) = A \cup B = \{1, 3, \ldots, 2n\} \subseteq \{0, 1, 2, \ldots, 2n\}$$

where $p = 2n + 1$. Hence $C_{2n+1}$ is edge - graceful.

Case (2): $L(C_{2n+1})$ is edge - graceful. $L(C_{2n+1})$ contains $2n + 1$ vertices and $2n + 1$ edges. Therefore by case (1) the graph $L(C_{2n+1})$ is edge - graceful. Hence the cycle $C_{2n+1}$ is bi-edge - graceful. The bi-edge - graceful labeling of $C_7$ is given in fig. 2.2.

**Fig. 2.2.** Bi-edge - graceful labeling of $C_7$

**Theorem**
The graph $C_{2n}$ is not bi-edge - graceful.

**Proof**
Here the cycle $C_{2n}$ contains $2n$ vertices and $2n$ edges. Clearly $p = 2n$ and $q = 2n$. Therefore, it does not satisfy the Lo’s necessary condition. Hence $C_{2n}$ is not edge - graceful. Thus $C_{2n}$ is not bi-edge - graceful.

**Theorem**
The graph $K_{i,n}$ is bi-edge - graceful if $n = 4t$, $t \geq 1$.

**Proof**
First we present the proof of $K_{i,n}$ is edge - graceful.
Case (1): $K_{i,n}$ is edge - graceful if $n = 4t$, $t \geq 1$.
Let $v, v_1, v_2, v_3, \ldots, v_n$ be the vertices of $K_{i,n}$ and the edges $e_i$ are defined as follows (see fig.2.3):

$e_i = (v, v_j)$ for $1 \leq i \leq n$

**Fig. 2.3.** $K_{i,n}$ with ordinary labeling

We now label the edges as follows:

$f(e_i) = i$ for $1 \leq i \leq n$

Then the induced vertex labels are

$$f^+(v_i) = 0$$

$$f^+(v_i) = i \quad \text{for} \quad 1 \leq i \leq n$$

These labels are arranged in order. Let

$$A = \{f^+(v_i) \mid 1 \leq i \leq n\} = \{0, 1, 2, \ldots, n\}$$

$$B = \{f^+(v_i) \mid n + 1 \leq i \leq 2n + 1\} = \{1, 2, 3, \ldots, n\}$$

$$f^+(V) = A \cup B = \{0, 1, 2, \ldots, n\}$$

Hence $K_{i,n}$ is edge - graceful if $n = 4t$, $t \geq 1$.

Case (2): $L(K_{i,n})$ is edge - graceful if $n = 4t$, $t \geq 1$.

Let $v_1, v_2, v_3, \ldots, v_n$ be the vertices of $L(K_{i,n})$ and the edges are (See fig. 2.4)

$e_{(i+1)} = (v_1, v_{i+1})$ for $i = 1, 2, 3, \ldots, n - 1$

$e_{(i,2)} = (v_1, v_{i+1})$ for $i = 1, 2, 3, \ldots, n - 2$

$e_{(i,3)} = (v_1, v_{i+1})$ for $i = 1, 2, 3, \ldots, n - 3$

$e_{2n} = (v_1, v_n)$

**Fig. 2.4.** $L(K_{i,n})$ with ordinary labeling

We now label the edges as follows:

For $1 \leq i \leq n - 1, \quad i + 1 \leq j \leq n$

$$f(e_{ij}) = i + \frac{(j - i - 1)(2n - j + 1)}{2}$$

Then the induced vertex labels are

$$f^+(v_i) = n - i \quad \text{for} \quad 1 \leq i \leq n - 1 \quad i \text{ odd}$$

$$f^+(v_i) = \frac{n}{2} - i \quad \text{for} \quad 2 \leq i \leq \frac{n}{2} \quad i \text{ even}$$

$$f^+(v_i) = \frac{3n}{2} - i \quad \text{for} \quad \frac{n}{2} + 2 \leq i \leq n \quad i \text{ even}$$

These labels are distinct. Hence $L(K_{i,n})$ is edge - graceful if $n = 4t$, $t \geq 1$.

By case (1) and case (2) the graph $K_{i,n}$ is bi-edge - graceful for $n = 4t$, $t \geq 1$. The edge - graceful labeling of $L(K_{i,8})$ is given in fig.2.5.
**Global edge-graceful**

A graph $G$ is called global edge-graceful if both $G$ and its complement $G'$ are edge-graceful.

**3.2. Theorem**

The graph $C_n$ $(n \geq 3)$ is global edge-graceful if $n$ is prime.

**Proof**

Case 1: $C_n$ is edge-graceful.

Let $v_1, v_2, ..., v_n$ be the vertices of $C_n$ and the edges $e_i$ are defined as follows:

- $e_i = (v_i, v_{i+1})$ for $1 \leq i \leq n-1$
- $e_n = (v_n, v_1)$ (see fig. 3.1)

By Theorem 2.3, $C_n$ is edge-graceful.

Case 2: $C_n^c$ is edge-graceful.

Let the vertices are defined as in case 1 and the edges are defined as follows:

- $e_{ij} = (v_i, v_j), \quad i + 2 \leq j \leq n-1$ for $2 \leq i \leq n-2$
- $e_{ij} = (v_i, v_j), \quad i + 2 \leq j \leq n$

**3.3 Definition**

A graph $G$ is called odd order global edge-graceful graphs if

$$f^+(v_i) = n - 2 - 3i$$ for $1 \leq i \leq \frac{n-2}{3}$

$$f^+(v_i) = 2n - 2 - 3i$$ for $\frac{n+1}{3} \leq i \leq \frac{2n-4}{3}$

$$f^+(v_i) = 3n - 2 - 3i$$ for $\frac{2n-1}{3} \leq i \leq n-1$

$$f^+(v_i) = 3i$$ for $i = n$

For $n = 7, 13, 19, 31, ...$

For $n = 5, 11, 17, 23, 29, ...$

These labels are distinct. Hence the graph $C_n^c$ is edge-graceful. Then by case 1 and case 2 $C_n$ is global edge-graceful graph.

The edge-graceful labeling of $C_n^c$ is given in fig. 3.3.

**Fig. 3.1 C_n with ordinary labeling**

**Fig. 3.2 C_n^c with ordinary labeling**

We now label the edges as follows (see fig. 3.2)

- For $i = 1 & i + 2 \leq j \leq n-1$
  $$f(e_{ij}) = 1 + \frac{(j-i-2)(2n-j)}{2}$$

- For $2 \leq i \leq n-2, i+2 \leq j \leq n$
  $$f(e_{ij}) = i + \frac{(j-i-2)(2n-j+i-1)}{2}$$

Then the induced vertex labels are

For $n = 5, 11, 17, 23, 29, ...$

For $n = 7, 13, 19, 31, ...$

By Theorem 2.3, $C_n$ is edge-graceful.

If a $(p, q)$ graph $G$ is global edge-graceful of odd order then $q \equiv 0 \pmod{p}$

**Proof**

If $G$ is a $(p, q)$ graph then $G'$ is a graph. Let $G$ be a global edge-graceful of odd order. Applying Lo’s condition we have

$$q(q+1) \equiv \frac{p(p+1)}{2} \pmod{p} \quad \text{... (1)}$$

Applying Lo’s condition for $G' (p, q_i)$ we have

$$q_1(q_1+1) \equiv \frac{p(p+1)}{2} \pmod{p} \quad \text{... (2)}$$

From equation (1) & (2) we have, 

$$q(q+1) \equiv q_1(q_1+1) \pmod{p}$$

$$\Rightarrow q(q+1) \equiv \frac{p-1}{2} - q \equiv \frac{p-1}{2} - q + 1 \pmod{p} \quad \text{... (3)}$$

Since $p$ is odd, equation (3) reduces to 

$$q(q+1) \equiv q' - q \pmod{p}$$

which implies $q \equiv 0 \pmod{p}$ since $p$ is odd.

**Theorem**

The odd order global edge-graceful graphs are precisely $(p, lp)$ graphs, where $l \leq l \leq \frac{P-1}{2}$. 

**Fig. 2.5. Edge-graceful labeling of $L(K_{1,n})$**

**Fig. 3.3 Edge-graceful labeling of $C_n^c$**
Proof

If a \((p,q)\) graph \(G\) is odd order global edge - graceful for some \(q \equiv 0 \pmod{p}\), \(1 \leq i \leq \frac{p-1}{2}\). Then one can easily verify that \(q \equiv i \pmod{p}\) for some \(i\), \(1 \leq i \leq p-1\). This implies that \(q \not\equiv 0 \pmod{p}\), a contradiction to the statement of Theorem 3.3.

Theorem

If a \((p,q)\) graph \(G\) is an odd order tree then \(G\) is not a global edge – graceful graph.

Proof

Suppose \(G\) is a global edge – graceful tree then by Theorem 3.3, \(q \equiv 0 \pmod{p}\), a contradiction to \(q = p-1\).

Example 1

Consider a \((6, 9)\) graph. Clearly, \(q \equiv 0 \pmod{p}\). But \(q \equiv 156 \equiv 0 \pmod{6}\) and \(\frac{p(p+1)}{2} = 21 \equiv 3 \pmod{6}\). So, by Lo’s necessary condition, \((6, 9)\) graph is not edge – graceful. Hence it is not a global edge – graceful graph.

Example 2

Consider a \((6, 12)\) graph. Clearly, \(q \equiv 0 \pmod{6}\). But \(q \equiv 1 \pmod{12}\) and \(\frac{p(p+1)}{2} = 21 \equiv 3 \pmod{12}\). So, by Lo’s necessary condition, \((6, 12)\) graph is not edge – graceful. Hence it is not a global edge – graceful graph.

Theorem

Any \((p,q)\) graph \(G\) where \(p \equiv 4t - 2\), \(t \geq 1\) is not global edge – graceful.

Proof

Suppose \(G\) with \(p = 4t - 2\), \(t \geq 1\) is global edge – graceful then by Theorem 3.6, we have \(q \equiv 0 \pmod{p}\) or \(\frac{p}{2} \pmod{p}\).

Case 1: \(q \equiv 0 \pmod{p}\)

Then, \(q = sp\) for some \(s\).

So, \(q + 1 = sp + (sp+1) \equiv 0 \pmod{p}\) and \(\frac{p(p+1)}{2} = (4t-2)(4t-1) = 4t-1\) (mod 12).

\(= 8t^2 - 6t + 1 = (4t - 2) 2t + (1 - 2t) \equiv \frac{p}{2} \pmod{p}\).

Therefore, \(q + 1 \equiv \frac{p(p+1)}{2} \pmod{p}\). So, by Lo’s condition \(G\) is not edge – graceful. Hence, \(G\) is not global edge – graceful.

Case 2: \(q \equiv \frac{p}{2} \pmod{p}\)

Then, \(q = sp + \frac{p}{2}\) for some \(s\).

So, \(q + 1 = sp + \frac{p}{2} + 1 \equiv \frac{p}{2} \pmod{p}\).

\(= sp + (sp+1) \equiv 0 \pmod{p}\).

Here, \(sp + \frac{p}{2} + 1\) is even and so \(q + 1 \equiv 0 \pmod{p}\). As in case 1, \(q + 1 \equiv \frac{p(p+1)}{2} \pmod{p}\). So by Lo’s condition, \(G\) is not edge – graceful. Hence, \(G\) is not global edge – graceful.

We now examine the global edge - gracefulness of a \((p,q)\) graph where \(p\) is of the form \(4t\), \(t \geq 1\). We split this into 4 cases namely \(p \equiv 4 \pmod{16}\), \(p \equiv 12 \pmod{16}\), \(p \equiv 0 \pmod{16}\) and \(p \equiv 8 \pmod{16}\). Fortunately, \(p \equiv 0 \pmod{8}\) or \(8 \pmod{16}\) can be considered as \(p \equiv 0 \pmod{8}\) and hence the number of cases boils down to 3.

Further we observe that in all the above cases, the global edge – gracefulness depends on the following condition.

\(q + 1 \equiv \frac{p(p+1)}{2} \pmod{p}\).

On these facts, we have some observations.

Observation

If \(p \equiv 4 \pmod{16}\) then \(q \equiv \frac{p}{4} \pmod{p}\) if and only if

\(q + 1 \equiv \frac{p(p+1)}{2} \pmod{p}\).

Proof

Assume \(q \equiv \frac{p}{4} \pmod{p}\). Then we have,

\(q+1 \equiv \frac{p}{4} + 1 \pmod{p}\).
Now, \( q(q+1) = \frac{p(p+1)}{2} \equiv k \pmod{p}. \) as \( p \equiv 4 \pmod{16}. \)

Thus, \( q(q+1) \equiv \frac{p(p+1)}{2} \pmod{p}. \)

Again, \( q_1 = \frac{p(p-1)}{2} - q \equiv \frac{p^2 - p - 2q}{2} \equiv \frac{p(p-1)}{2} \equiv \frac{p(p+1)}{2} \pmod{p}. \)

As proved above, we can see that \( q_1(q_1+1) \equiv \frac{p(p+1)}{2} \pmod{p}. \)

Thus we have, \( q(q+1) \equiv q_1(q_1+1) \equiv \frac{p(p+1)}{2} \pmod{p}. \)

Conversely, Assume that \( q(q+1) \equiv \frac{p(p+1)}{2} \pmod{p} \)

\( q(q+1) \equiv \frac{p(p+1)}{2} \pmod{p} \Rightarrow q(q+1) = lp + \frac{p}{2} \)

or \( q = \frac{1}{2}(1 + \sqrt{1 + 4(lp + \frac{p}{2})}) \).

Since \( q \) is an integer, choose \( l \) such that \( 1 + 4(lp + \frac{p}{2}) = s^2. \)

As \( q > 0, \) \( q = \frac{s-1}{2}. \)

Similarly, \( q_1(q_1+1) \equiv \frac{p(p+1)}{2} \pmod{p} \) implies \( q_1 = \frac{s-1}{2}. \) So, \( q = q_1. \)

Again, \( q_1 = \frac{p(p-1)}{2} - q \Rightarrow 2q = \frac{p(p-1)}{2} \equiv \frac{p(p+1)}{2} \pmod{p}. \)

Therefore, \( q = \frac{p}{4} \pmod{p}. \)

**Observation**

If \( p \equiv 4 \pmod{16} \) then \( q \equiv \frac{3p}{4} \pmod{p} \) if and only if

\( q(q+1) \equiv q_1(q_1+1) \equiv \frac{p(p+1)}{2} \pmod{p}. \)

**Proof**

Assume \( q = \frac{3p}{4} \pmod{p} \). Then we have,

\( q+1 \equiv \frac{3p}{4} + 1 \pmod{p}. \)

Now,

\( q(q+1) = \frac{3p}{4} \cdot \frac{3p+1}{4} - \frac{p}{2} \equiv \frac{p(9p+4)}{16} \equiv 0 \pmod{p}. \)

\( p = 12 \pmod{16} \Rightarrow 9p + 4 \equiv 0 \pmod{16}. \)

So, \( q(q+1) \equiv \frac{p}{2} \pmod{p}. \)

Again, \( q_1 = \frac{p(p-1)}{2} - q \equiv \frac{p(p-1)}{2} - \frac{3p}{4} \pmod{p} \)

\( \equiv \frac{p^2 - 5p}{4} \pmod{p} = \frac{p(2p-5)}{4} \equiv 0 \pmod{p}. \)

Therefore, \( q_1 \equiv 12 \pmod{16}. \)

Therefore, \( q_1 = \frac{3p}{4} \pmod{p}. \)

By the arguments discussed for \( q \), one can prove that \( q_1(q_1+1) \equiv \frac{p}{2} \pmod{p}. \)

By following the proof of observation 3.9, one can prove the converse by similar lines.

**Theorem**

If a \((p,q)\) graph \( G \) is global edge – graceful where \( p \equiv 4 \pmod{16} \) then \( q \equiv \frac{p}{4} \pmod{p}. \)

**Proof**

Suppose \( G \) is global edge – graceful. Then by Lo’s condition we have, \( q(q+1) \equiv q_1(q_1+1) \equiv \frac{p(p+1)}{2} \pmod{p}. \) Now by observation 3.9, \( q \equiv \frac{p}{4} \pmod{p}. \)

**Theorem**

If a \((p,q)\) graph \( G \) is global edge – graceful where \( p \equiv 12 \pmod{16} \) then \( q \equiv \frac{3p}{4} \pmod{p}. \)

**Proof**

Follows from Lo’s condition and observation 3.10.

**Theorem**

If \( p \equiv 4 \pmod{16} \) then the global edge – graceful graphs are precisely \( \left\{p, \frac{p}{2} + lp \right\} \) graphs where \( 1 \leq l \leq \frac{p-4}{2}. \)

**Proof**

Proof follows from Theorem 3.11 and by following the arguments of Theorem 3.4.

**Theorem**

If \( p \equiv 2 \pmod{16} \) then the global edge – graceful are precisely \( \left\{p, \frac{3p}{4} + lp \right\} \) graphs where \( 1 \leq l \leq \frac{p-6}{2}. \)

**Proof**

Proof follows from Theorem 3.12 and by following the arguments of Theorem 3.4.

**Theorem**

Any \((p,q)\) graph \( G \) with \( p \equiv 0 \pmod{8} \) is not global edge – graceful

**Proof**

**Case 1:** \( q \equiv \frac{p}{2} \pmod{p} \)

Then, \( q(q+1) \equiv \frac{p}{2} \pmod{p} \).

But, \( q_1 = \frac{p(p-1)}{2} - q \equiv \frac{p(p-1)}{2} - \frac{p}{2} \pmod{p} \)

\( \equiv \frac{p(p-2)}{2} \pmod{p} \).

Therefore, \( G^c \) is not edge – graceful. Thus, \( G \) is not global edge – graceful.

**Case 2:** \( q \equiv \frac{p}{2} - 1 \pmod{p} \)

Then, \( q(q+1) \equiv \left( \frac{p}{2} - 1 \right) \pmod{p} \).

(Since \( p \equiv 0 \pmod{8} \).)
But, \( q_1 = \frac{p(p-1)}{2} - q = \frac{p(p-1)}{2} - \frac{p+1}{2} + \frac{p-2}{2} + 1 \equiv \lfloor mp \rfloor \).

Now, \( q_1 + 1 \equiv 2 \pmod{p} \).

Therefore, \( G' \) is not edge-graceful. Thus, \( G \) is not global edge-graceful.

Case 3: Neither \( p \equiv q \pmod{2} \) nor \( p \equiv q^{-1} \pmod{2} \).

In this case, one can prove that \( q(q+1) \not\equiv \frac{p(p+1)}{2} \pmod{p} \).

which shows that \( G \) is not edge-graceful. Thus, \( G \) is not global edge-graceful.

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