Introduction

By a graph $G = (V,E)$ we mean a finite undirected graph without loops or multiple edges. Terms not defined have are used in the sense of Harary[1].

A set $S \subseteq E$ is said to be an edge dominating set if every edge in $E - S$ is adjacent to some edge in $S$. The edge domination number of $G$ is the cardinality of a smallest edge dominating set of $G$ and is denoted by $\gamma'$. The degree of an edge $e = uv$ of $G$ is defined by $\text{deg } e = \text{deg } u + \text{deg } v - 2$. The minimum (maximum) degree of an edge in $G$ is denoted by $\delta (\Delta )$. For a real no $x$, $[x]$ denotes the largest integer $\leq x$ and $\lceil x \rceil$ denotes the smallest integer $\geq x$. We need the following theorems.

Theorem 1.1[2] If $G$ is a graph with $p \geq 3$ then $\gamma' \leq \left\lfloor \frac{2p}{3} \right\rfloor$.

Theorem 1.2[2] (i) If $G$ is a graph without isolated vertices, then $\gamma' \leq p - \Delta + 1$.

(ii) If $G$ is connected and $\Delta = p - 1$, then $\gamma' \leq p - \Delta$.

Theorem 1.3[2] For any $(p,q)$ - graph $G$ with isolated vertices, $d_1 \leq \min (\delta, p/\gamma')$.

Main results

Definition 2.1 Let $G = (V,E)$ be a graph without isolated edges. An edge dominating set $X$ of $G$ is called a total edge dominating set if the edge induced subgraph $\langle X \rangle$ has no isolated edges. The minimum cardinality of a total edge dominating set is called the total edge domination number of $G$ and is denoted by $\gamma'_t (G)$ or $\gamma'_t$. The upper total edge domination number of $G$ is the maximum cardinality taken over all minimal total edge dominating sets of $G$ and is denoted by $\Gamma_t'.

Example 2.2

(i) $\gamma'_t (P_n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \text{ or } 2 \text{ (mod 4)} \\ \frac{n-1}{2} & \text{if } n \equiv 1 \text{ (mod 4)} \\ \frac{n+1}{2} & \text{if } n \equiv 3 \text{ (mod 4)} \end{cases}$

(ii) $\gamma'_t (C_n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \text{ (mod 4)} \\ \frac{n+1}{2} & \text{if } n \equiv 1 \text{ or } 3 \text{ (mod 4)} \\ \frac{n+2}{2} & \text{if } n \equiv 2 \text{ (mod 4)} \end{cases}$

(iii) $\gamma'_t (K_{1,n}) = 2$

(iv) $\gamma'_t (K_{m,n}) = \min \{m,n\}$ if $m,n > 1$.

Theorem 2.3 $\gamma'_t (K_p) = \left\lfloor \frac{2p}{3} \right\rfloor$ if $p \geq 3$.

Proof Let $V(K_p) = \{v_1,v_2,\ldots,v_p\}$. Let $C_i = \{e_i \mid 1 \leq i \leq p-1\}$. If $p \equiv 0 \text{ or } 1 \text{ (mod 3)}$, let $D = \{e_i \mid 1 \leq i \leq p-1 \text{ and } i \not\equiv 0 \text{ (mod 3)}\} \bigcup \{e_{p-2}\}$.
Clearly $D$ is a total edge dominating set of $K_p$ and $|D| = \left\lceil \frac{2p}{3} \right\rceil$. Hence $\gamma'_e(K_p) \leq \left\lfloor \frac{2p}{3} \right\rfloor$.

Now let $D$ be any total edge dominating set of $K_p$. If there exist two vertices, say $v_i, v_j$ of $K_p$ which are not incident with any edge of $D$, then the edge $v_iv_j$ is not dominated by $D$. Hence $D$ must cover at least $p-1$ vertices of $K_p$. Further $\langle D \rangle$ has no isolated edges and hence $|D| \geq \left\lfloor \frac{2(p-1)}{3} \right\rfloor = \left\lfloor \frac{2p}{3} \right\rfloor$.

Thus $\gamma'_e(K_p) = \left\lfloor \frac{2p}{3} \right\rfloor$.

Theorem 2.4 $\gamma'_e(W_p) = \left\lfloor \frac{p}{2} \right\rfloor$.

Proof Let $V(W_p) = \{v_1, v_2, \ldots, v_p\}$, deg $v_i = p-1$ and $E(W_p) = \{v_1v_i | 2 \leq i \leq p\} \bigcup \{v_2v_3, v_3v_4, \ldots, v_{p-1}v_p, v_pv_p\}$. Then $S = \{v_1v_2v_3v_4, \ldots, v_{p-1}v_p, v_pv_1\}$ if $p$ is even \{v_1v_2v_3v_4, \ldots, v_{p-1}, v_p\}$ if $p$ is odd is a total edge dominating set of $W_p$ so that $\gamma'_e(W_p) \geq \left\lfloor \frac{p}{2} \right\rfloor$.

Now let $S$ be a minimum total edge dominating set of $W_p$. Since any two adjacent edges $e_1, e_2$ of $S$ dominate at most four edges of $C_p = (v_2v_3, \ldots, v_pv_2)$ in $W_p$ (including possibly $e_1$ and $e_2$), it follows that $|S| \geq \left\lfloor \frac{p}{2} \right\rfloor$. Hence $\gamma'_e(W_p) = \left\lfloor \frac{p}{2} \right\rfloor$.

Remark 2.5 If $G$ is a graph without isolated edges, then $\gamma'_e(G) = \gamma'_e(L(G))$ where $L(G)$ denotes the line graph of $G$. Hence it follows from Theorems 1.1 and 1.2 that

(i) $\gamma'_e \leq \left\lfloor \frac{2q}{3} \right\rfloor$

(ii) $\gamma'_e \leq q - \Delta' + 1$

(iii) If $G$ is connected and $\Delta' < q - 1$ then $\gamma'_e \leq q - \Delta'$

For the graphs given in Figure 4.1 $\gamma'_e = 4 = \left\lfloor \frac{2q}{3} \right\rfloor$.

Figure 2.1

Theorem 2.6 For any tree $T$ with $\Delta' < p - 2$, $\gamma'_e \leq q - \Delta'$ if and only if $4 \leq \text{diam}(T) \leq 6$ and $T$ is isomorphic to one of the following.

(1) $P_4, P_5, P_6$ or $P_7$.

(2) Any tree with exactly one vertex $u$ of degree $\geq 3$ satisfying the following condition. If there exists a pendant vertex $x$ with $d(u,x) = 4$, then there must exist at most one pendant vertex $y$ with $d(u,y) = 2$.

(3) Any tree obtained from a tree described in (2) by attaching any number of pendant vertices to exactly one vertex $v$ of $N(u)$ such that in the resulting tree $v$ has degree $\geq 3$.

Proof Let $T$ be a tree with $\Delta' < p - 2$. Suppose $4 \leq \text{diam}(T) \leq 6$ and $T$ is isomorphic to one of the trees given in the hypothesis. Then $|S| = \Delta'$ where $S$ is the set of all pendant vertices of $E(G) \setminus S$ is the unique minimum total edge dominating set of $T$ so that $\gamma'_e \leq q - \Delta'$.

Conversely suppose that $\gamma'_e \leq q - \Delta'$. Then diam$(T) \geq 4$ and $|S| = \Delta'$. Since $E(T)S$ is the unique minimum total edge dominating set of $T$, it follows that diam$(T) \leq 6$ and $T$ has at most one vertex of degree $\geq 3$ or two adjacent vertices of degree $\geq 3$. We consider the following cases.

Case (i) $T$ has no vertex of degree $\geq 3$.

In this case, $T = P_5, P_6$ or $P_7$.

Case (ii) $T$ has exactly one vertex, say $u$ of degree $\geq 3$.

Suppose there exists a pendant vertex $x$ in $T$ such that $d(u,x) = 4$. Let $P = (u, u_1, u_2, u_3, u_4, x)$ be the $u - x$ path of length 4. If there exist two pendant vertices, say $y_1, y_2$ such that $d(u,y_1) = d(u,y_2) = 2$ then $E(T) \setminus \{S \cup \{ux_1\}\}$ is a total edge dominating set of $T$ so that $\gamma'_e \leq q - \Delta' - 1 < q - \Delta'$ which is a contradiction. Hence there is at most one pendant vertex $y$ with $d(u,y) = 2$.

Case (iii) $T$ has exactly two adjacent vertices, say $u, v$ of degree $\geq 3$.

First we claim that there exist at least deg$(u) - 2$ pendant vertices adjacent to $v$. Otherwise there exist vertices $u_1, u_2, v_1, v_2$ such that $d(u_1) = d(u_2) = d(v_1) = d(v_2) = 2$ and in this case $E(T) \setminus \{S \cup \{uv\}\}$ is a total edge dominating set of $T$ with cardinality $q - \Delta' - 1$, which is a contradiction. Hence we assume that there exist deg$(u) - 2$ pendant vertices adjacent to $u$ in $T$. Let $T_1$ be the tree obtained by deleting these deg $u - 2$ pendant vertices adjacent to $u$. Clearly $T_1$ is a tree with exactly one vertex $v$ of degree $\geq 3$ and $\gamma'_v = q(T_2) - \Delta(T_1)$.

Hence $T_1$ is of the form described in (2) and the result follows.

Notation 2.7 (i) Let $G$ be a graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$. Let $k_1, k_2, \ldots, k_m$ be non negative integers. Take $k_i$ copies of $P_{1,i-1} = 1, 2, \ldots, n$ where $P_{1,i}$ denotes the path on $i$ vertices. Then the graph obtained from $G$ by identifying one end vertex of each $P_i$ with $v_1$ is denoted by $G*_{v_1}(k_1P_{1,1}, k_2P_{1,2}, \ldots, k_mP_{1,m})$.

If such paths are attached at more than one vertex of $G$, the above notation can be extended in an obvious way.

(ii) Let $C_n$ be the cycle $(v_1, v_2, \ldots, v_n, v_1)$. The graph obtained by attaching a pendant edge, say $wv_1$, to $C_n$ is denoted by $E_{w}$. Thus $E_{w} = C_n * v_1(P_2)$.

Example 2.8 Let $G = C_3 = (v_1, v_2, v_3, v_4)$. Then $G * v_1(2P_2, 2P_2, P_2) * v_2(2P_2, P_2) * v_3(3P_2)$ and $E_{w} = v_1(2P_2, P_2, P_2) * w(3P_2)$ are given in Figure 2.1.
Theorem 2.9 Let G be a connected unicyclic graph with cycle $C_n = (v_1, v_2, ..., v_n, v_1)$. Then $\Delta' \geq q - \Delta'$ if and only if G is isomorphic to one of the following

1. $C_n = v_1(a_{11}P_2) \cup v_2(a_{21}P_2)$ where $a_{11}, a_{21} \geq 0$.
2. $C_n = v_1(b_1P_2) \cup v_2(b_2P_2)$ where $b_{11}, b_{21} \geq 0$.
3. $E_1 = v_1(c_{12}P_2) \cup v_2(c_{22}P_2)$ where $c_{11}, c_{21} \geq 0$.
4. $E_2 = v_1(d_{12}P_2) \cup v_2(d_{22}P_2)$ where $d_{11}, d_{21} \geq 0$.
5. $E_3 = v_1(e_{12}P_2) \cup v_2(e_{22}P_2)$ where $e_{11}, e_{21} \geq 0$.
6. $E_4 = v_1(f_{12}P_2) \cup v_2(f_{22}P_2)$ where $f_{11}, f_{21} \geq 0$.
7. $E_5 = v_1(g_{12}P_2) \cup v_2(g_{22}P_2)$ where $g_{11}, g_{21} \geq 0$.
8. $E_6 = v_1(h_{12}P_2) \cup v_2(h_{22}P_2)$ where $h_{11}, h_{21} \geq 0$.
9. $E_7 = v_1(l_{12}P_2) \cup v_2(l_{22}P_2)$ where $l_{11}, l_{21} \geq 0$.
10. $E_8 = v_1(m_{12}P_2) \cup v_2(m_{22}P_2)$ where $m_{11}, m_{21} \geq 0$.
11. $E_9 = v_1(n_{12}P_2) \cup v_2(n_{22}P_2)$ where $n_{11}, n_{21} \geq 0$.
12. $E_{10} = v_1(q_{12}P_2) \cup v_2(q_{22}P_2)$ where $q_{11}, q_{21} \geq 0$.

Proof Let G be a connected unicyclic graph with cycle $C = C_n = (v_1, v_2, ..., v_n, v_1)$. Let $e = uv$ be an edge of maximum degree $\Delta'$ and let S denote the set of all pendant edges of G. Clearly $|S| \geq \Delta' - 2$.

Now suppose $q \leq q - \Delta'$. Since $E(G) \setminus (S \cup \{e\})$ where $e$ is any edge of C is a total edge dominating set of G, it follows that $|S| = \Delta' - 1$ or $\Delta' - 2$. Hence G has at most three vertices of degree $\geq 3$.

If there exist two non adjacent vertices $w_1, w_2$ with deg $w_1 \geq 3$ and deg $w_2 \geq 3$, then $|S| = \Delta' - 1$ and in this case there exist two adjacent edges $e_1, e_2$ of C such that $E(G) \setminus (S \cup \{e_1, e_2\})$ is a total edge dominating set of cardinality $q - \Delta' - 1$, which is a contradiction. Thus any two vertices of degree $\geq 3$ are adjacent.

Hence if G has three vertices of degree $\geq 3$, then these three vertices form a triangle and hence C = $C_3$ and in this case at least one vertex of $C_3$ has degree exactly 3.

We now claim that the edge $e = uv$ of maximum degree lies on C or incident with a vertex of C. If both u and v are not on C, then $|S| = \Delta' - 1$ and there exist two adjacent edges $e_1, e_2$ of C such that $E(G) \setminus (S \cup \{e_1, e_2\})$ is a total edge dominating set of cardinality $q - \Delta' - 1$, which is a contradiction. Hence at least one of u, v lies on C.

Now suppose $4 \leq n \leq 6$. Then G has at most two vertices of degree $\geq 3$ and $|S| = \Delta' - 2$. If G has no vertex of degree $\geq 3$, then G isomorphic to one of the graphs given in (1), (2) or (4).

We now assume that G has at least one vertex of degree $\geq 3$. Without loss of generally we assume that $\deg v_1 \geq 3$ and $\deg v_2 = 2$. Then $D = E(G) \setminus (S \cup \{v_1v_2, v_2v_3\})$ is a minimum total edge dominating set of cardinality $q - \Delta'$. We now consider the following cases.

Case(i) $C = C_6$.

If G contains an induced subgraph H isomorphic to the graph $C_6 \ast v_1(P_3)$, then $D \setminus \{v_3v_6\}$ is a total edge dominating set of G, which is a contradiction. Hence G is isomorphic to the graph given in (1)

Case(ii) $C = C_4$.

If G contains an induced subgraph H isomorphic to the graph $C_4 \ast v_1(P_3)$, then $D \setminus \{v_3v_4\}$ is a total edge dominating set of G, which is a contradiction. Therefore distance of every vertex from $C_4$ is at most 2.

Subcase (a) $e = v_1v_5$.

Let $e = v_1v_5$. If G contains an induced subgraph H isomorphic to $C_4 * v_1(P_3) * v_5(P_3)$, then $D \setminus \{v_1v_5\}$ is a total edge dominating set of G, which is a contradiction. Hence G is isomorphic to the graph given in (2).

Subcase (b) $e = v_1v_5$.

Let $e = v_1v_5$. Since distance of any vertex from $C_4$ is at most 2, all the vertices adjacent to w are all pendant vertices. Since by Subcase (a), all the vertices adjacent to $v_1$ other than w are all pendant vertices, G is isomorphic to the graph given in (3).

Case(iii) $C = C_4$.

If G contains an induced subgraph H isomorphic to the graph $C_4 * v_1(P_3)$. Then $D \setminus \{v_1v_4\}$ where $u_1$ is a vertex of H adjacent to $v_4$ in H is a total edge dominating set, which is a contradiction. Hence all the vertices not on $C_4$ are at distance of at most 3 from $C_4$. 

![Figure 2.1](image-url)
Suppose $G$ contains an induced subgraph $H$ isomorphic to $C_3 \star v_1(P_3) \star v_2(P_2)$ or $C_3 \star v_1(P_3) \star v_2(P_2)$, and $D \cup \{e\}$ is a total edge dominating set of $G$, which is a contradiction. Hence $G$ is isomorphic to the graph given in (7).

Case (iv) $C_3 \in C_3$.

Suppose $G$ contains an induced subgraph $H$ isomorphic to $C_3 \star v_1(P_3)$ or $C_3 \star v_2(P_2)$ or $C_3 \star v_3(P_3)$. If $H = C_3 \star v_1(P_3)$, then $E(G) \setminus (S \cup \{v_1(P_3), v_1(P_3), v_1(P_3)\})$ is a total edge dominating set of $S(G)$, which is a contradiction. Hence there is at most one path of length four which is edge disjoint from $C_3$ in $G$.

If $|S| = \Delta' - 1$, then $D = E(G) \setminus (S \cup \{v_1(P_3)\})$ is a minimum total edge dominating set of $G$ with cardinality $q - \Delta'$. If $|S| = \Delta' - 2$, then $G$ contains a vertex of $C_3$, say $v_3$, of degree 2 and $D_1 = E(G) \setminus (S \cup \{v_1(P_3), v_2(P_2)\})$ is a minimum total edge dominating set of $G$ with cardinality $q - \Delta'$.

Subcase (a) $e$ lies on $C_4$.

Let $e = v_1v_2$. If $G$ contains an induced subgraph $H$ isomorphic to $C_4 \star v_1(P_3) \star v_2(P_2)$ or $C_4 \star v_1(P_3) \star v_2(P_2)$, then $D \cup \{e\}$ is a total edge dominating set of $G$, which is a contradiction. Hence $G$ is isomorphic to the graph given in (4),(5) or (6).

Subcase (b) $e$ is incident with a vertex of $C_4$.

Let $e = v_1v_2$. If $G$ contains an induced subgraph $H$ isomorphic to $C_4 \star v_1(P_3) \star v_2(P_2)$, then $D \cup \{e\}$ is a total edge dominating set of $G$, which is a contradiction. Hence $G$ is isomorphic to the graph given in (7).

Theorem 2.10 For any graph $G$ of order $p$, $\gamma'(S(G)) \leq 2(p - \beta_1)$.

Proof Let $X = \{u_i, v_i | 1 \leq i \leq \beta_1\}$ be a maximum edge independent set of $G$. Then $X$ is an edge dominating set of $G$. Let $b_i$ be the vertex of $S(G)$ which is adjacent to both $u_i$ and $v_i$. Let $S$ be the set of vertices of $G$ which are not incident with any edge of $X$. If $S = \emptyset$ then $D = \{u_1w_1, v_1w_2, u_2w_2, v_2w_2, \ldots, u_{\beta_1}w_{\beta_1}, v_{\beta_1}w_{\beta_1}\}$ is a total edge dominating set of $S(G)$ so that $\gamma'(S(G)) \leq 2\beta_1 = 2(p - \beta_1)$. Suppose $S \neq \emptyset$. Let $S = \{x_1, x_2, \ldots, x_n\}$. Since $G$ is connected and $\langle S \rangle$ is independent, each $x_i$ is adjacent to some $z_i$ $(z_i = u_i$ or $v_i)$. Let $y_i$ be the vertex of $S(G)$ adjacent to both $x_i$ and $z_i$ in $S(G)$. Then $D_2 = \{u_1w_1, v_1w_2, u_2w_2, v_2w_2, \ldots, u_{\beta_1}w_{\beta_1}, v_{\beta_1}w_{\beta_1}, x_1y_1, x_2y_2, \ldots, x_{\beta_1}y_{\beta_1}\}$ forms a total edge dominating set of $S(G)$ so that $\gamma'(S(G)) \leq |D_2| = 2\beta_1 + 2 = 2(\beta_1 + 2) = 2(p - \beta_1)$.

The inequality in Theorem 2.10 cannot be improved further. In fact equality holds for $K_p$ and $K_m,n$ as shown in the following theorem.

Theorem 2.11

1. $\gamma'(S(K_p)) = 2 \left\lceil \frac{p}{2} \right\rceil$.

2. $\gamma'(S(K_{m,n})) = 2n$ ($m \leq n$).

Proof (1) Since $\beta_1(K_p) = \left\lceil \frac{p}{2} \right\rceil$ by Theorem 2.10, it follows that $\gamma'(S(K_p)) \leq 2 - \left\lceil \frac{p}{2} \right\rceil$. To prove the reverse inequality, let $D$ be a total edge dominating set for $S(K_p)$. Suppose there exist two vertices, say $u,v$, such that neither $u$ nor $v$ is incident with any edge of $D$. Let $w$ be adjacent to both $u$ and $v$ in $S(K_p)$. Now the edges $uw$ and $vw$ are not dominated by $D$ so that $D$ is not a total edge dominating set of $S(K_p)$. Hence $D$ must cover at least $p - 2$ vertices of $K_p$. Further $D$ has no isolated edges and hence it follows that $|D| \geq 2 - \left\lceil \frac{p}{2} \right\rceil$. Thus $\gamma'(S(K_p)) = 2 \left\lceil \frac{p}{2} \right\rceil$.

(2) Let $(X,Y)$ be the bipartition of $K_{m,n}$ with $|X| = m$ and $|Y| = n$. By Theorem 2.10, we have $\gamma'(S(K_{m,n})) \leq 2(p - \beta_1) = 2(m + n) - 2m = 2n$. Further any total edge dominating set for $S(K_{m,n})$ must contain at least $2n$ edges for dominating the edges incident with the vertices of $X$ and $m$ of the vertices of $Y$ and $2(m-n)$ edges for dominating the edges incident with the remaining $n-m$ vertices. Hence $|D| \geq 2m + 2(n - m) = 2n$. Thus $\gamma'(S(K_{m,n})) = 2n$. 


Theorem 2.12 Let T be any tree of order \( p \geq 3 \) and \( n \) be the number of pendant edges of T.

1. \( n \leq \gamma'_v(S(T)) \leq 2p - 2 - n \).
2. \( \gamma'_v(S(T)) = 2p - 2 - n \) if and only if T is isomorphic to \( K_{1,n} \) or \( P_2 \).
3. \( \gamma'_v(S(T)) = n \) if and only if any internal vertex of T is adjacent to at least two pendant vertices.

Proof (1) Let \( u_1,v_1,u_2,v_2,\ldots, u_n,v_n \) be the pendant edges of T such that \( \deg_T v_1 = 1 \). Let \( w_i \) be the vertex of S(T) that subdivides the edge \( u_i,v_i \). Any total edge dominating set of S(T) contains the edges \( u_iw_i, i = 1,2,\ldots,n \) and hence \( \gamma'_v(S(T)) \geq n \). Further \( E(S(T)) \cup S \) where S is the set of all pendant edges of S(T) forms a total edge dominating set of S(T) and hence \( \gamma'_v(S(T)) \leq 2p - 2 - n \).

(2) Suppose \( \gamma'_v(S(T)) \leq 2p - 2 - n \). We claim that \( \text{diam}(T) \leq 3 \). Suppose \( \text{diam}(T) \geq 4 \). Let \( P = (v_1, v_2, \ldots, v_k, v_{k+1}) \) be path of length k in T. Let \( w_i \) be the vertex of S(T) subdividing the edge \( v_i v_{i+1}, 1 \leq i \leq k \). Now \( E(S(T)) \setminus (S \cup \{w_2v_1\}) \) is a total edge dominating set of cardinality \( 2p - 2 - n - 1 \), which is a contradiction. Hence \( \text{diam}(T) \leq 3 \). If \( \text{diam}(T)=2 \) then T is isomorphic to \( K_{1,n} \).

Suppose \( \text{diam}(T) = 3 \). Let \( P_2 = (u_1, u_2, u_3, u_4) \) be a path of length 3 in T. We claim that \( \text{T} = P_2 \). Suppose \( \text{deg} u_2 \geq 3 \). Since \( \text{diam}(T) = 3 \) any vertex \( w \neq u_2 \) is adjacent to \( u_2 \) is a pendant vertex of T. Let \( N(u_2) = \{u_2, w_1, w_2, \ldots, w_k\} \). Let x be a vertex subdividing the edge \( u_2x \). Now \( E(S(T)) \setminus (S \cup \{u_2x\}) \) is a total edge dominating set of cardinality \( 2p - 2 - n - 1 \), which is a contradiction. Hence \( \text{deg} u_2 = 2 \). Similarly \( \text{deg} u_3 = 2 \) and hence T is isomorphic to \( P_2 \).

Conversely, if \( T = P_2 \) then \( \gamma'_v(S(T)) = 4 = 2p - 2 - n \) and if \( T = K_{1,n} \) then \( \gamma'_v(S(T)) = p - 1 = 2p - 2 - n \).

(3) Suppose \( \gamma'_v(S(T)) = n \). Let \( u_1, v_1, u_2, v_2, \ldots, u_n, v_n \) be the pendant edges of T such that \( \deg_T v_1 = 1 \). Let \( w_i \) be the vertex of S(T) that subdivides the edge \( u_i, v_i \). Since any total edge dominating set of S(T) contains the edges \( u_iw_i \) for all \( i = 1,2,\ldots,n \) and \( \gamma'_v = n, S = \{u_iw_i | i = 1,2,\ldots,n\} \) is the unique minimum total edge dominating set of S(T). The edge induced subgraph \( \langle S \rangle \) is isomorphic to a union of stars \( K_{1,n_i} \), with \( n_i \geq 1 \) and every non-pendant edge of T joins the centers of two such stars. Hence any internal vertex of T is adjacent to at least two pendant vertices.

The converse is obvious.

Definition 2.13 The total edge domatic number of G, denoted by \( d'_T(G) \) or \( d' \), is the maximum order of a partition of the edge set E into total edge dominating sets of G.

Example 2.14

(i) \( d'_T(P_n) = 1 \).

(ii) \( d'_T(C_n) = \begin{cases} 2 \text{ if } n \equiv 0 \text{ (mod 4)} \\ 1 \text{ otherwise.} \end{cases} \)

(iii) \( d'_T(K_{1,n}) = \left\lfloor n/2 \right\rfloor \).

(iv) \( d'_T(K_{m,n}) = \max \{m,n\} \) if \( m,n \geq 2 \).

Remark 2.15 Since for any graph G, \( d'_T(G) = d'_T(L(G)) \), by Theorem 1.3 it follows that for any \( (p,q) \)-graph G without isolated edges, \( d'_T(G) \leq \min \{\delta', q/\gamma'_v \} \).

Theorem 2.16

(1) For any \( (p,q) \)-graph without isolated edges, \( \gamma'_v + d'_T \leq q + 1 \) and equality holds if and only if \( G = mP_1 \).

(2) If G is connected and \( p \geq 4 \), then \( \gamma'_v + d'_T \leq q \) and equality holds if and only if \( G = C_4, K_{1,4}, K_{1,3} \) or \( P_4 \).

Proof

(1) By Remarks 2.5 and 2.15, we have \( \gamma'_v \leq q - \Delta' + 1 \) and \( d'_T \leq \Delta' \). Hence \( \gamma'_v + d'_T \leq q - \Delta' + 1 + \Delta' \leq q + 1 \). Further \( \gamma'_v + d'_T = q + 1 \) if and only if \( \gamma'_v = q - \Delta' + 1 \) and \( \Delta' = 0 \). Since \( d'_T \leq \frac{q}{\gamma'_v} \) and \( \gamma'_v + d'_T = q + 1 \), it follows that \( d'_T \leq \frac{q}{\gamma'_v} \) which implies \( (q - d'_T)(d'_T - 1) \leq 0 \).

Since \( q - d'_T > 0 \), we have \( d'_T = 1 = \delta' \) and hence \( G = P_1 \).

(2) By(1), \( P_1 \) is the only connected graph with \( \gamma'_v + d'_T \leq q + 1 \) and hence \( \gamma'_v + d'_T \leq q \) for any connected graph with \( p \geq 4 \).

Suppose \( \gamma'_v + d'_T = q \). We consider the following cases.

Case(i) \( \Delta' \leq q - 1 \)

Then \( \gamma'_v = q - \Delta' + 1, d'_T = \delta' \) and \( \Delta' = 0 \). Since \( d'_T \leq \frac{q}{\gamma'_v} \), we have \( d'_T \leq \frac{q}{q - d'_T} \) so that \( d'_T \geq q(d'_T - 1) \). Further \( q \geq 2d'_T \) and hence \( d'_T \geq 2d'_T(d'_T - 1) \) so that \( d'_T \leq 2 \). If \( d'_T = 1 = \delta' \) then \( G = P_3 \), which is a contradiction since \( p \geq 4 \). If \( d'_T = 2 = \delta' \) then \( G = C_4 \).

Case (ii) \( \Delta' = q - 1 \)

Then \( \gamma'_v = 2 \) and \( d'_T = q - 2 \). Now since \( d'_T \leq \frac{q}{\gamma'_v} \), we have \( q - 2 \leq \frac{q}{\gamma'_v} = \frac{q}{2} \) so that \( q \leq 4 \).

If \( q = 3 \), then \( G = P_4 \) or \( K_{1,3} \). If \( q = 4 \), then \( G = K_{1,4} \). Thus \( G \) is isomorphic to \( C_4, K_{1,4}, K_{1,3} \) or \( P_4 \).

The converse is obvious.

References