Controllability results for impulsive differential systems with infinite delay

S. Selvi\(^1\) and M. Mallika Arjunan\(^2\)

\(^1\)Department of Mathematics, Muthayamal College of Arts & Science, Rasipuram- 637408, Tamil Nadu, India.
\(^2\)Department of Mathematics, Karunya University, Karunya Nagar, Coimbatore- 641 114, Tamil Nadu, India.

**ABSTRACT**

This paper is concerned with the controllability of impulsive functional differential equations with infinite delay in a Banach space. Sufficient conditions for controllability are obtained by using the measures of noncompactness and Monch fixed point therem under the assumption of noncompactness of the evolution system.

**Keywords**

Controllability, Impulsive differential equations, Measures of noncompactness, Semigroup theory, Fixed point.

**Introduction**

The theory of impulsive differential equations has been emerging as an important area of investigation in recent years, because, all the structures of its emergence have deep physical background and realistic mathematical models. The theory of impulsive differential equations appears as a natural description of several real processes subject to certain perturbations whose duration is negligible in comparison with the duration of the process. It has seen considerable development in the last decade, see the monographs of Bainov and Simeonov [2], Lakshmikantham et al. [13] and Samoilenko and Perestyuk [22] and the papers of [1, 4, 7, 8, 18, 24, 25], where the numerous properties of their solutions are studied and detailed bibliographies are given. In many areas of science there has been an increasing interest in the investigation of functional differential equations, incorporating memory or aftereffect, i.e., there is an effect of infinite delay on state equations. We refer the reader to Kolmanovskii and Myshkis [11, 12], Wu [23] and the references there in for a wealth of reference materials on the subject.

The notion of controllability is of great importance in mathematical control theory. Many basic problems of control theory pole-assignment, structural engineering and optimal control may be solved under the assumption that the system is controllable. The concept of controllability plays an crucial role in both finite and infinite dimensional spaces, that is systems represented by ordinary differential equations and partial differential equations, respectively. In recent years, significant progress has been made in the controllability of linear and nonlinear deterministic systems [5, 6, 9, 14, 15, 17, 19, 21]. In [9], the author studied the controllability of impulsive functional differential systems of the form

\[
x(t) = A(t)x(t) + f(t, x(t)) + B(t)x(t), \quad a.e. \quad \text{on } [0, b],
\]

\[
A(t)x \rightarrow X, \quad t = 1, 2, ..., \infty,
\]

where \(X\) is a family of linear operators which generates an evolution operator

\[
U(t) = \{(x, s) \in [0, b] \times [0, b] : 0 \leq s \leq t \leq b \} \rightarrow \mathcal{L}(X),
\]

here \(X\) is a Banach space, \(\mathcal{L}(X)\) is the space of all bounded linear operators in \(X\), \(f : [0, b] \times X \rightarrow X \), \(t \in [0, b], t \neq t_0, k = 1, 2, ..., m\), are impulsive functions;

\[
M: \mathcal{L}(X) \rightarrow \mathcal{L}(X); \quad B \text{ is a bounded linear operator from a Banach space } V \text{ to } X \text{ and the control function } u(t) \text{ is given in } L^2([0, b], V).
\]

The results are obtained by using the measures of noncompactness and Monch fixed point theorem.

Inspired by the above mentioned works [5, 9, 25], in this paper we investigate the sufficient conditions for the controllability of impulsive functional system with infinite delay of the form

\[
x(t) = A(t)x(t) + f(t, x(t)) + B(t)x(t), \quad a.e. \quad \text{on } [0, b],
\]

\[
A(t)x \rightarrow X, \quad t = 1, 2, ..., \infty,
\]

where \(B(t)\) is the abstract phase space which will be defined later; \(A(t)\) is a family of linear operators which generates an evolution system \(U(t)\); \(f : [0, b] \times X \rightarrow X\); \(t \in [0, b], t \neq t_0, k = 1, 2, ..., m\); \(x(t)\) is the state variable; \(X\) takes the values in the real Banach space \(X\) with norm \(||\cdot||\); The control function \(u(t)\) is given in \(L^2([0, b], V)\) a Banach space of admissible control functions with \(V\) as a Banach space; \(B(t)\) is a bounded linear operator from \(V\) into \(X\); \(f : [0, b] \times X \rightarrow X\) is given function; \(A(t)\) is a family of linear operators which generates an evolution system \(U(t)\); \(t \in [0, b], t \neq t_0, k = 1, 2, ..., m\); \(x(t)\) and \(x(t)\) represent the right and left limits of \(x(t)\) at \(t = t_0\), respectively. Denote

\[
A(t) = \{(x, s) \in [0, b] \times [0, b] : 0 \leq s \leq t \leq b \} \rightarrow \mathcal{L}(X),
\]

where \(x(t)\) is a continuous at \(t = t_0\) and left continuous at \(t = t_0\) and the right limit \(x(t)\) exists for \(k = 1, 2, ..., m\). It is easy to verify that

\[
\mathcal{P}_{C}((-\infty, b], X) = \{x : (-\infty, b] \rightarrow X \text{ such that } x(t) \text{ is continuous at } t = t_0 \text{ and left continuous at } t = t_0 \text{ and the right limit } x(t) \text{ exists for } k = 1, 2, ..., m\}.
\]

Let \(\mathcal{P}_{C}((-\infty, b], X) = \{x : (-\infty, b] \rightarrow X\} \text{ be a Banach space with the norm } ||x||_{\mathcal{P}_{C}} = \sup \{||x(t)|| \leq t | t \in (-\infty, b]\} \)
The histories \( x_t^{(a)}(\cdot, 0, \cdot) \to X \) defined by
\[
x_t^{(a)}(\cdot) = x(t + a), \quad -\infty < a \leq 0,
\]
which belong to \( \mathbb{B}_2 \). Our approach here is based on semigroup theory, measures of noncompactness and Monch fixed point theorem.

### Preliminaries

At first, we define the abstract phase space \( \mathbb{B}_2 \) as given in [5].

Assume that \( h: (-\infty, 0) \to (0, +\infty) \) is a continuous function with \( h(r) \to +\infty \) as \( r \to -\infty \). For any \( a \geq 0 \), we define and equip the space \( \mathbb{B}_2 \) with the norm
\[
\| x \|_{\mathbb{B}_2} = \sup_{t \geq 0} \{ \int_t^{t_a} | x(s) | ds + h(t) \}, \quad x \in \mathbb{B}_2.
\]

Let us define
\[
\mathbb{B}_2 = \{ x \in FC((-\infty, 0), X) : x(t) = \phi, \quad t \leq 0 \} \subset X\text{ such that } x(t) \text{ is bounded and measurable for } t > 0.
\]

### Definition 2.1

A function \( x \in FC((-\infty, 0), X) \) is said to be a mild solution of the system (1.1)–(1.3) if, for every initial function \( \phi \in \mathbb{B}_2 \) and for any \( t \geq 0 \) there exists a control \( u \in L^1(L^p(X)) \) such that the mild solution \( x(t) \) of (1.1)–(1.3) satisfies \( x(t) = \phi \).

### Definition 2.2

The system (1.1)–(1.3) is said to be controllable on the interval \( I \) if, for every initial function \( \phi \in \mathbb{B}_2 \) and for any \( t \geq 0 \) there exists a control \( u \in L^1(L^p(X)) \) such that the mild solution \( x(t) \) of (1.1)–(1.3) satisfies \( x(t) = \phi \).

### Definition 2.3

Let \( \mathbb{E}^n \) be the positive cone of an order Banach space \( (E, \leq) \). A function \( \mathcal{G} \) defined on the set of all bounded subsets of the Banach space \( X \) with values in \( \mathbb{E}^n \) is called a measure of noncompactness (MNC) on \( X \) if \( \mathcal{G}(x(t)) = \mathcal{G}(t) \) for all bounded subsets \( \mathcal{G} \subset X \), where \( \mathcal{G} \) stands for the closed convex hull of \( \mathcal{G} \).

The MNC \( \mathcal{G} \) is said:

- Monotone if for all bounded subsets \( \mathcal{G}_1, \mathcal{G}_2 \) of \( X \) we have:
  \[
  \mathcal{G}_1 \subset \mathcal{G}_2 \Longleftrightarrow \mathcal{G}(\mathcal{G}_1) \subset \mathcal{G}(\mathcal{G}_2);
  \]

- Nonsingular if \( \mathcal{G}(x(t)) = \mathcal{G}(x(t), x(t)) \) for every \( a \leq t \in X, \mathcal{G} \subset X \);

- Regular if \( \mathcal{G}(t) = 0 \) if and only if \( \mathcal{G} \) is relatively compact in \( X \).

One of the most examples of MNC is the noncompactness measure of Hausdorff \( \mathcal{G} \) defined on each bounded subset \( \mathcal{G} \) of \( X \) by
\[
\mathcal{G}(\mathcal{G}) = \lim_{r \to 0} \mathcal{G}(Q_r); \quad \mathcal{G} \text{ can be covered by a finite number of balls of radii smaller than } a.
\]

It is well known that MNC \( \mathcal{G} \) enjoys the above properties and other properties see [3, 10].

### Further Definitions

#### Definition 2.4

A two parameter family of bounded linear operators \( U(t, s) \) on \( X \) is said to be a bounded evolution system if the following two conditions are satisfied:

(i) \( U(t, s) = I \) for \( 0 \leq s \leq t \leq b \);

(ii) \( U(t, s) \) is strongly continuous for \( 0 \leq s \leq t \leq b \).

### Lemma 2.1

(15) Assume that \( x \in \mathbb{B}_2 \); then for \( t \leq s \in \mathcal{G} \);

Moreover,
\[
\begin{align*}
\| x(t) \|_{\mathbb{B}_2} & \leq \| x(s) \|_{\mathbb{B}_2} + \int_t^s \| x(u) \|_{\mathbb{B}_2} du + \int_t^s h(u) \| x(u) \|_{\mathbb{B}_2} du.
\end{align*}
\]

### Lemma 2.2

(3) If \( W \subset C([a, b], X) \) is bounded and equicontinuous, then \( \mathcal{G}(W) \) is continuous for \( t \in [a, b] \) and

### Lemma 2.3

(25) If \( W \subset FC([-\infty, 0], X) \) is bounded and piecewise equicontinuous on \( I \) then \( \mathcal{G}(W) \) is piecewise continuous for \( t \in [a, b] \) and

### Lemma 2.4

(17) Let \( (f_n) \) be a sequence of functions in \( L^1(L^p(X)) \). Assume that there exist \( \mathcal{G} \) satisfying \( \mathcal{G}(f_n) \leq \mathcal{G}(f) \) and then for all \( t \in [a, b] \), we have
\[
\int_a^b \mathcal{G}(f_n(t)) dt \leq \int_a^b \mathcal{G}(f(t)) dt.
\]

### Lemma 2.5

(17) Let \( (f_n) \) be a sequence of functions in \( L^1(L^p(X)) \). Assume that there exist \( \mathcal{G} \) satisfying \( \mathcal{G}(f_n) \leq \mathcal{G}(f) \) and then for all \( t \in [a, b] \), we have
\[
\int_a^b \mathcal{G}(f_n(t)) dt \leq \int_a^b \mathcal{G}(f(t)) dt.
\]

The following fixed-point theorem, a nonlinear alternative of Monch type, plays a key role in our proof of controllability of the system (1.1)–(1.3).

### Lemma 2.6

(16, Theorem 2.2) Let \( D \) be a closed convex subset of a Banach space \( X \) and \( 0 \in D \). Assume that \( F: D \to X \) is a continuous map which satisfies Monch’s condition, that is
\[
\mathcal{G}(F(t)) \leq \mathcal{G}(t, 0) \quad \text{for all } t \leq 0;
\]

Then \( F \) has a fixed point in \( D \).

### Controllability Results

In this section, we present and prove the controllability results for the problem (1.1)–(1.3). In order to prove the main theorem of this section, we list the following hypotheses:

- (H1). \( A(t) \) is a family of linear operators, \( A(t)A(0) = X \) not depending on \( t \) and dense subset of \( X \); generating an equicontinuous evolution system \( \{ U(t, s) : 0 \leq t \leq s \leq b \} \); i.e., \( \mathcal{G}(t, 0) \) is equicontinuous for \( t > 0 \) and for all bounded subsets \( \mathcal{G} \) and

- (H2). The function \( f: \mathcal{G} \to X \) satisfies:

For a.e. \( t \in J \), the function \( f(t) \) is continuous and for all \( \phi \in \mathcal{G} \), the function \( f(t, \phi) \) is strongly measurable.
For every positive integer \( r \), there exists \( \alpha_r \in L^1(0, b; \mathbb{R}^n) \) such that
\[
\sup_{s \in [0, b]} \| f(t, s, x(t)) \| \leq \alpha_r(t), \quad t \in \mathbb{R}^n,
\]
and
\[
\lim_{s \to b} \int_0^s f(t, s, x(t)) \, ds = \sigma < \infty.
\]

There exists integrable function \( \beta \in L^1(0, b; \mathbb{R}^n) \) such that
\[
\beta(t) < \sup_{t \in [0, b]} \| f(t, s, x(t)) \| < \beta(t) + \alpha_r(t), \quad t \in [0, b],
\]
and
\[
\lim_{s \to b} \int_0^s \beta(t) \, ds = \int_0^b \beta(t) \, dt = D < \infty.
\]

(H3). The linear operator \( \mathcal{W} : L^1(0, b; \mathbb{R}^n) \to X \) is defined by
\[
\mathcal{W}(y) = \int_0^b U(t, s) B(\alpha) \, ds,
\]
where \( D = \{ y(0) : y \in \mathbb{R}^n \} \).

\( \mathcal{W} \) has an invertible operator \( W^{-1} \), which takes values in \( L^1(0, b; \mathbb{R}^n) \), and there exist positive constants \( M_2 \) and \( M_3 \) such that
\[
\| \mathcal{W}^{-1} \| \leq M_2, \quad \| W^{-1} \| \leq M_3.
\]

There is \( \mathcal{K}_1 \in L^1(0, b; \mathbb{R}^n) \) such that, for every bounded set \( Q \subset X \),
\[
\beta(W^{-1})(Q) \leq \mathcal{K}_w(Q) \beta(Q),
\]
and
\[
\lim_{s \to b} \int_0^s \beta(t) \, ds = \mathcal{K}_w < \infty, \quad k = 1, 2,..., m.
\]

(H4). Let \( \mathcal{L} \) be a continuous operator such that: There are nondecreasing functions \( \alpha_2 : \mathbb{R}^n \to \mathbb{R}^n \) such that
\[
\| \mathcal{L} \| \leq \alpha_2, \quad x \in \mathbb{R}^n.
\]

There exist constants \( c_k \geq 0 \) such that,
\[
\beta((\mathcal{L})(x)) \leq \alpha_2(x), \quad k = 1, 2,..., m,
\]
and
\[
\lim_{s \to b} \int_0^s \beta(t) \, ds = c_k < \infty, \quad k = 1, 2,..., m.
\]

(H5). The following estimation holds true:
\[
\| \mathcal{L} \| \leq \alpha_2, \quad x \in \mathbb{R}^n.
\]

**Theorem 3.1.** Assume that the hypotheses (H1) – (H5) are satisfied. Then the impulsive differential system (1.1) – (1.3) is controllable on \( J \), provided that
\[
M_k(1 + M_2 M_3 b^3 + \mathcal{K}_w b^3) \leq \frac{1}{c}.
\]

**Proof.** Using the hypothesis (H5), for every \( x \in X \), we define the control
\[
\phi(t) = \mathcal{L}(x(t)), \quad t \in J.
\]

We shall now show that when using this control the operator defined by
\[
\tilde{\mathcal{G}}(x) = U(t, s)[f(x(t)) + \mathcal{G}(\phi(t)) + \mathcal{L}(x(t)) + B(\alpha)t] \, ds + \int_0^t \left[ \sum_{k=1}^m \Delta \phi_k(t) \delta(t-s) \right] \, ds
\]
has a fixed point. This fixed point is then a solution of \( \tilde{\mathcal{G}}(x) = x \). Clearly \( x(t) = \mathcal{L}(x(t)) \) which implies the system \( \tilde{\mathcal{G}}(x) = x \) is controllable. We rewrite the problem as follows:

For \( \phi \in B_b \), we define
\[
\phi(t) = \mathcal{L}(x(t)), \quad t \in J.
\]

Then \( \phi \in B_b \). Let \( x(t) = \gamma(t) + \phi(t), \quad t \in (-\infty, 0) \). It is easy to see that \( \gamma(t) \) satisfies
\[
x(t) = U(t, s) [f(x(t)) + \mathcal{G}(\phi(t)) + \mathcal{L}(x(t)) + B(\alpha)] \, ds + \sum_{k=1}^m \Delta \phi_k(t) \delta(t-s), \quad t \in J.
\]

Define
\[
x(t) = \phi(t), \quad t \in (-\infty, 0).
\]

and
\[
x(t) = \phi(t), \quad t \in [0, b].
\]

and thus \( \mathcal{G}(\phi(t)) \) is a Banach space. Set
\[
\mathcal{E}_q = \{ y(t) = \phi(t), y(t) = \phi(t) \}.
\]

Clearly \( \mathcal{E}_q \) is a nonempty, closed, convex and bounded set in \( 
\mathcal{E}_q \). Then for any \( y \in \mathcal{E}_q \), from Lemma 2.1, we have
\[
\| y(t) + \phi(t) \| \leq \| y(t) + \phi(t) \| + \| \phi(t) \| = \| y(t) + \phi(t) \| + \| \phi(t) \|.
\]

In the view of Lemma 2.1, for each \( t \in J \),
\[
\sup_{t \in [0, b]} \| y(t) + \phi(t) \| \leq \| y(t) + \phi(t) \| + \| \phi(t) \|.
\]

Let \( \mathcal{G} : \mathcal{E}_q \to \mathcal{E}_q \) be an operator defined by
\[
\mathcal{G}(x)(t) = \mathcal{L}(x(t)) + \mathcal{L}(x(t)) + B(\alpha)
\]
Obviously the operator \( \mathcal{G} \) has a fixed point is equivalent to \( \mathcal{G} \) has one. So it turns out to prove \( \mathcal{G} \) has a fixed point. Let \( \mathcal{G} = \mathcal{G}_1 + \mathcal{G}_2 \).

\[
\mathcal{G}(y)(t) = \mathcal{L}(x(t)) + \mathcal{L}(x(t)) + B(\alpha)
\]
(3.5)

\[
(\mathcal{G}(y)(t) = \mathcal{L}(x(t)) + \mathcal{L}(x(t)) + B(\alpha))
\]
(3.6)

**Step 1:** There exists a positive number \( \alpha_2 \) such that
\[
\| \mathcal{G}(x)(t) \| \leq \alpha_2, \quad x \in \mathcal{E}_q.
\]

Suppose the contrary. Then for each positive integer \( q \), there exists a function \( y^q(t) \in \mathcal{E}_q \) but \( \| \mathcal{G}(y^q)(t) \| \geq q \) for some \( t \in J \). We have from
\[
\| (\mathcal{G}(y^q)(t) = \mathcal{L}(x(t)) + \mathcal{L}(x(t)) + B(\alpha))
\]
(3.7)

\[
\| \mathcal{G}(y^q)(t) \| \leq \| \mathcal{G}(y^q)(t) \| + \| \phi(t) \|,
\]

where
\[
\| \mathcal{G}(y^q)(t) \| \leq \| \mathcal{G}(y^q)(t) \| + \| \phi(t) \|.
\]

Hence by
\[
\| y^q \| \leq \| y^q \| + \| \phi(t) \| = \| y^q \| + \| \phi(t) \|,
\]

\[
\| y^q \| \leq \| y^q \| + \| \phi(t) \|,
\]

where \( M = \| \mathcal{G}(x)(t) \| + \| \phi(t) \| \) is independent of \( q \). Dividing both sides by \( q \) and noting that \( q = \infty \), we obtain

\[
\| y^q \| \leq \| y^q \| + \| \phi(t) \| = \| y^q \| + \| \phi(t) \|,
\]

we obtain
Thus we have

\[
1 \leq M_1 \frac{(1 + R_j M_j) \mu(t) + \sum_{i \in A_j} \lambda_i}{M_2}
\]

This contradicts (3.1). Hence for some positive number \(q\),

**Step 2:** \(B : B_j \to B_0\) is continuous.

Let \(\{f^{(n)}(t)\}_{n=0}^{\infty} \to f(t)\) as \(n \to \infty\). In the view of (3.2), we have

\[
\|y^{(n)}(t) - y(t)\| \leq q, \quad t \in J.
\]

Then there is a number \(q \geq 0\) such that \(\|y^{(n)}(t)\| \leq q\) for all \(n \to \infty\) and \(t \in J\). By \(B_j(t)\), for each \(t \in J\), By (3.3),

\[
f(c, y^{(n)}(t) + \phi_n(t)) - f(c, y^{(n)}(t)) = f(c, y^{(n)}(t) + \phi_n(t)) - f(c, y^{(n)}(t)) + f(c, y^{(n)}(t)) - f(c, y(t)) + f(c, y(t)) - f(c, y^{(n)}(t))
\]

Then we have

\[
\|G_j y^{(n)} - G_j y\| \leq M_1 \int_J \|f(c, y^{(n)} + \phi_n) - f(c, y^{(n)} + \phi_0)\| + M_2 \int_J (\|y^{(n)} - y\| + \|\phi_0 - \phi_n\|)
\]

where

\[
\|y^{(n)} - y\| \leq M_1 \int_J \|f(c, y^{(n)} + \phi_0) - f(c, y^{(n)} + \phi_0)\| + M_2 \int_J (\|y^{(n)} - y\| + \|\phi_0 - \phi_n\|)
\]

Observe that (3.10) and by dominated convergence theorems we have that.

\[
\|G_j y^{(n)} - G_j y\| \to 0 \quad \text{as} \quad n \to \infty.
\]

That is \(G\) is continuous.

**Step 3:** \(G\) is equicontinuous on the \(f_j\), \(k = 1, 2, \ldots, m\). That is \(G(B_j)\) is piecewise equicontinuous on \(J\).

Indeed for \(t_2, t_3 \in f_j, t_2 < t_3\) and \(y \in B_j\), we deduce that

\[
G_j(y^{(n)}(t_2)) - G_j(y^{(n)}(t_3)) \leq M_1 \int_J \|f(c, y^{(n)}(t_2) + \phi_n(t)) - f(c, y^{(n)}(t_3) + \phi_n(t))\| + M_2 \int_J (\|y^{(n)}(t_2) - y^{(n)}(t_3)\| + \|\phi_n(t_2) - \phi_n(t_3)\|)
\]

By the equicontinuity of \(G_j(y^{(n)}(t))\) and the absolute continuity of the Lebesgue integral, we can see that the right hand side of (3.12) tends to zero and independent of \(y \in B_j\). Hence \(G_0(B_j)\) is equicontinuous on \(J_0\).

**Step 4:** The Monch’s condition holds.

Suppose \(W \subseteq B_0\) is countable and \(W \subseteq \Xi(x) \cup G(W)\). From (3.13), we obtain that

\[
\beta \left(\left(\bigcup_{n=1}^{m} G_j( f_j)\right) \Delta \bigcup_{n=1}^{m} G_j( f_j)\right) = 0, \quad \text{where} \quad \beta = \lambda.
\]

Without loss of generality, we may assume that \(W = (y^{(n)})_{n=1}^{\infty}\). Since \(G\) maps \(W\) into an equicontinuous family, \(G(W)\) is equicontinuous on \(J_0\). Hence \(G_0(B_j)\) is also equicontinuous on \(J_0\).

By (3.3), we have

\[
\beta \left(\left(\bigcup_{n=1}^{m} G_j( f_j)\right) \Delta \bigcup_{n=1}^{m} G_j( f_j)\right) = 0.
\]

By Lemma 2.4 and from (3.7) and (3.8), we have that

\[
\beta \left(\left(\bigcup_{n=1}^{m} G_j( f_j)\right) \Delta \bigcup_{n=1}^{m} G_j( f_j)\right) = 0.
\]

This implies that

\[
\beta \left(\left(\bigcup_{n=1}^{m} G_j( f_j)\right) \Delta \bigcup_{n=1}^{m} G_j( f_j)\right) = 0.
\]

From (3.13) and (3.14) we obtain that

\[
\beta \left(\left(\bigcup_{n=1}^{m} G_j( f_j)\right) \Delta \bigcup_{n=1}^{m} G_j( f_j)\right) = 0.
\]

for each \(t \in J\). Since \(W\) and \(G(W)\) are equicontinuous on \(J_0\) for every \(J_0\), according to Lemma 2.3, the inequality (3.15) implies that,

\[
\beta \left(\left(\bigcup_{n=1}^{m} G_j( f_j)\right) \Delta \bigcup_{n=1}^{m} G_j( f_j)\right) = 0.
\]


That is \( (\Phi(t)W) \leq \Phi(t)(\Phi(t)W) \leq \Phi(t)(\Phi(t)W) = 0 \), \( W \) is defined in (H5). Thus from the Monch's codition, we get that
\[
N \leq 1 \leq \frac{1}{N},
\]
which implies that \( N \) is relatively compact in \( E_k \). In the view of Lemma 2.6, i.e., Monch's fixed point theorem, we conclude that \( \Phi(t)W \) has a fixed point \( \gamma \) in \( W \). Then \( \gamma = \gamma + 0 = \gamma \) is a fixed point of \( \Phi(t) \) and thus the system \( (1.1) - (1.2) \) is controllable on \([0, b] \).

**Remark 3.1.** Note that if \( f \) is compact or Lipschitz continuous, then \( \Phi(t)(\Phi(t)W) \) is automatically satisfied. In the following, by using another MNC, we will prove the result of the Theorem 3.1, in the case there is no equivalence of the evolution system \( U(t, x) \) and hypothesis (H3). Here we assume that the impulsive operators \( I_0 \) are compact. So, instead of \( (H4) \), we give the hypothesis
\[
(H4') \quad \exists I_0 = R^n \to R^n \text{ such that there are nondecreasing functions } I_k : R^+ \to R^+ \text{ satisfying}
\]
\[
\sup_{x \in X} I_k(t) \leq I_k(0) = 1, \quad k = 1, 2, ..., m.
\]

**Theorem 3.2.** Let \( \{A(t)\}_{t \in [0, b]} \) be a family of linear operators that generates a strongly continuous evolution system \( \{U(t, x)\}_{t \in [0, b]} \). Assume that the hypothesis \( (H1), (H2), (H3) \) and \( (H4') \) are satisfied. Then the impulsive differential system
\[
(1.1) - (1.3)
\]
is controllable on \([0, b] \).

**Proof.** In the view of Theorem 3.1, we should only prove that the function \( G : E_k \to E_k \) given by the formula (3.4) satisfies the Monch's condition.

For this purpose, let \( W \leq E_k \) be countable and \( \mathcal{B}_k \). We shall prove that \( W \) is relatively compact.

We will denote by \( \Phi \) the following MNC in \( E_k \) defined by (see[10]),
\[
\Phi(t)W = \left\{ \sum_{i=1}^{n} \alpha_i E_i(t) \right\}
\]
for all bounded subsets of \( E_k \), where \( \{E_i(t)\} \) is the set of countable subsets of \( E_k \), \( \alpha \) is the real MNC defined by,
\[
\alpha(t) = \sup_{x \in E_k} e^{-\int_{0}^{t} \alpha(E(t))dt}.
\]

with \( E_k = \{x(t) : x \in E \} \), \( L \) is a constant that we shall choose appropriately.

\( \text{mod} \) is the modulus of equicontinuity of the function set \( E \) given by the formula
\[
\text{mod}(E) = \sup_{E \in E_k} \max_{t \in [0, b]} \left( \left| \frac{d}{dt} \|x(t)\|_{E_k} \right| \right)
\]
\( \text{mod} \) is defined in theorem (3.1) (H2). Let \( M_k = \text{mod}(E) \) as defined in theorem (3.1). From the regularity of \( \Phi \), it is enough to prove that \( \Phi(t)(\Phi(t)W) \) is a maximum, let \( \gamma(t) = \gamma(t) \) be the denumerable set which achieves its maximum. Then there exists a set \( \gamma(t) \), \( \gamma(t) \) such that
\[
(3.17)
\]
Now we give an estimation for \( \gamma(t) \). Since \( I_3(t) \) is compact, we get
\[
\gamma(t) = \gamma(t) \gamma(t) \gamma(t) \gamma(t) \gamma(t) \gamma(t) \gamma(t) \gamma(t) \gamma(t)
\]
From (3.14) and (3.15), noticing that \( C_i = 0 \) as \( C_i \) is compact, we have that
\[
\gamma(t) \gamma(t) \gamma(t) \gamma(t) \gamma(t) \gamma(t) \gamma(t) \gamma(t) \gamma(t)
\]
Therefore, we have that
\[
\gamma(t) \gamma(t) \gamma(t) \gamma(t) \gamma(t) \gamma(t) \gamma(t) \gamma(t) \gamma(t)
\]
From (3.18), we obtain that
\[
\gamma(t) \gamma(t) \gamma(t) \gamma(t) \gamma(t) \gamma(t) \gamma(t) \gamma(t) \gamma(t)
\]
From the definition of \( \alpha \), we have
\[
(3.17)
\]
From (3.14) and (3.15), noticing that \( C_i = 0 \) in (3.14), we get that
\[
(3.17)
\]
That is, \( \gamma(t) \) is relatively compact for almost all \( t \in [0, b] \) in \( E_k \). Moreover, from the fact that \( \gamma(t) \), \( \gamma(t) \), and \( \gamma(t) \), \( \gamma(t) \) is easy to see that \( \gamma(t) \), \( \gamma(t) \), \( \gamma(t) \), \( \gamma(t) \) is uniformly integrable for a.e. \( t \). So, \( \gamma(t) \), \( \gamma(t) \), \( \gamma(t) \), \( \gamma(t) \) is semicompact according to the Definition 2.3. By applying Lemma 2.5, we have that \( \gamma(t) \), \( \gamma(t) \), \( \gamma(t) \), \( \gamma(t) \) is relatively compact in \( E_k \).

On the other hand, by the strong continuity of \( \gamma(t) \), \( \gamma(t) \), \( \gamma(t) \), \( \gamma(t) \), we can easily verify that \( \gamma(t) \), \( \gamma(t) \), \( \gamma(t) \), \( \gamma(t) \) is relatively compact. Then by (3.19), \( \gamma(t) \), \( \gamma(t) \), \( \gamma(t) \), \( \gamma(t) \) is also relatively compact in \( E_k \). Since \( \Phi \) is a monotone, nonsingular, regular MNC, from Monch’s condition, we have that
\( \Phi(W) \leq \Phi(\overline{\mathcal{D}_0} \cup \mathcal{G}_1(W)) = \Phi([z_0,z_1]) = (0,0). \)

Therefore, \( W \) is relatively compact in \( \mathbb{H} \). This completes the proof.

References