Introduction


Preliminaries

Definition 2.1.[2] Let \( L = \{ L, \leq \} \) be complete lattice with an involutive order reversing operation \( N : L \rightarrow L \). Then an Intuitionistic L-fuzzy Subset (ILFS) \( A \) in a non-empty set \( X \) is defined as an object of the form 
\[
\{< x, \mu_A(x), \nu_A(x)> / x \in X \}
\]
where \( \mu_A : X \rightarrow L \) is the degree membership and \( \nu_A : X \rightarrow L \) is the degree non-membership of the element \( x \in X \) satisfying \( \mu_A(x) \leq N(\nu_A(x)) \).

Definition 2.2 [1] A BF-algebra is a non-empty set \( X \) with a consonant 0 and a binary operation \( \ast \) satisfying the following axioms:

i. \( x \ast x = 0 \)

ii. \( x \ast 0 = x \)

iii. \( 0 \ast (x \ast y) = y \ast x \ \forall \ x, y \in X \)

Definition 2.3.

(1) A non-empty subset \( S \) of a BF-algebra \( X \) is said to be a BF-subalgebra if 
\[
x \ast y \in S \ \forall x, y \in S.
\]

(2) A non-empty subset \( I \) of a BF-algebra \( X \) is said to be an Ideal of \( X \) if

\[
(i) \ 0 \in I \\
(ii) \ x \ast y \in I \ \text{and} \ y \in I \\
\text{imply that} \ x \in I \ \forall \ x, y \in I.
\]

(3) An ideal \( I \) of \( X \) is called closed 
\[
\text{if} \quad 0 \ast x \in I \ \forall x \in I.
\]

(4) A non-empty subset \( I \) of a BF-algebra \( X \) is said to be an H-ideal of \( X \)
\[
(i) \ 0 \in I \\
(ii) \ x \ast (y \ast z) \in I \ \text{and} \ y \in I \\
\text{imply that} \ x \ast z \in I \ \forall \ x, y, z \in I.
\]

(5) A H-ideal \( I \) of \( X \) is called closed if
\[
0 \ast x \in I \ \forall x \in I.
\]

Definition 2.4. Let \( \langle X, \ast_x, 0_x \rangle, \langle Y, \ast_y, 0_y \rangle \) be two BF-algebras. The Cartesian product of \( X \times Y \) is defined to be the set
\[
X \times Y = \{ (x, y) / x \in X, y \in Y \}
\]
In \( X \times Y \) we define the product \( x \times y \) as follows:
\[
(x_1, y_1) \ast_{x \times y} (x_2, y_2) = (x_1 \ast_x x_2, y_1 \ast_y y_2)
\]
One can easily verify that the Cartesian product of two BF-algebras is again a BF-algebra.

Definition 2.5.[6] An Intuitionistic L-fuzzy Subset \( A \) in a BF-algebra \( X \) is said to be an Intuitionistic L-fuzzy BF-ideal of \( X \) if

1) \( \mu_A(0) \geq \mu_A(x) \)
2) \( \nu_A(0) \leq \nu_A(x) \)
3) \( \mu_A(x) \geq \mu_A(x^*y) \land \mu_A(y) \).
4) \( \nu_A(x) \leq \nu_A(x^*y) \lor \nu_A(y) \quad \forall x, y \in X \)

**Definition 2.6** [6] An intuitionistic L-fuzzy subset \( A \) of a BF-algebra \( X \) is said to be an intuitionistic L-fuzzy closed BF-ideal of \( X \) if
1) \( \mu_A(0^*x) \geq \mu_A(x) \)
2) \( \nu_A(0^*x) \leq \nu_A(x) \)
3) \( \mu_A(x^*z) \geq \mu_A(x^*(y^*z)) \land \mu_A(y) \) 
4) \( \nu_A(x^*z) \leq \nu_A(x^*(y^*z)) \lor \nu_A(y) \quad \forall x, y, z \in X \)

**Definition 2.7** [8] An intuitionistic L-fuzzy subset \( A \) of a BF-algebra \( X \) is said to be an intuitionistic L-fuzzy H-ideal of \( X \) if
1) \( \mu_A(0) \geq \mu_A(x) \)
2) \( \nu_A(0) \leq \nu_A(x) \)
3) \( \mu_A(x^*z) \geq \mu_A(x^*(y^*z)) \land \mu_A(y) \) 
4) \( \nu_A(x^*z) \leq \nu_A(x^*(y^*z)) \lor \nu_A(y) \quad \forall x, y, z \in X \)

**Product on intuitionistic L-fuzzy H-ideals of BF-algebras**
In this section we introduce the notion of Cartesian Product of two intuitionistic L-fuzzy H-ideals of two BF-algebras \( X \) and \( Y \). We start with the following definition.

**Definition 3.1** [7] For any two intuitionistic L-fuzzy sets \( A \) and \( B \) of \( X \) and \( Y \), their Cartesian Product is defined to be the set \( A \times B = (X \times Y, \mu_A \times \mu_B, \nu_A \times \nu_B) \) with the membership and non-membership functions \( \mu_A \times \mu_B : X \times Y \rightarrow \mathbb{L} \) and \( \nu_A \times \nu_B : X \times Y \rightarrow \mathbb{L} \) such that

\[
\mu_A \times \mu_B(x, y) = \mu_A(x) \land \mu_B(y) \\
\nu_A \times \nu_B(x, y) = \nu_A(x) \lor \nu_B(y)
\]

where \( \forall x \in X \) and \( y \in Y \).

**Theorem 3.3**. Let \( A \) and \( B \) be any two intuitionistic L-fuzzy H-ideals of \( X \) and \( Y \). Then \( A \times B \) is an intuitionistic L-fuzzy H-ideal of \( X \times Y \).

**Proof**. Take \((x, y) \in X \times Y \).

Then \( (\mu_A \times \mu_B)(0, 0) = \mu_A(0) \land \mu_B(0) \geq \mu_A(x) \land \mu_B(y) \) 
and \( (\nu_A \times \nu_B)(0, 0) = \nu_A(0) \lor \nu_B(0) \leq \nu_A(x) \lor \nu_B(y) \)

Now take \((x_1, y_1) \), \((x_2, y_2) \) and \((x_3, y_3) \) \( \in X \times Y \).

Then \( (\mu_A \times \mu_B)[(x_1, y_1) \ast (x_3, y_3)] \)

\[
= (\mu_A \times \mu_B)[(x_1^* \times x_3^*), (y_1^* \times y_3^*)]
\]

\[
= (\mu_A(x_1) \land \mu_B(y_1) \land (\mu_A(x_3) \land \mu_B(y_3))
\]

\[
= (\mu_A(x_1) \land \mu_B(y_1) \lor (\mu_A(x_3) \land \mu_B(y_3))
\]

\[
= (\mu_A(x_1) \lor \mu_B(y_1) \land (\mu_A(x_3) \lor \mu_B(y_3))
\]

\[
= (\mu_A(x_1) \lor \mu_B(y_1) \lor (\mu_A(x_3) \lor \mu_B(y_3))
\]

Also \( (\nu_A \times \nu_B)[(x_1, y_1) \ast (x_3, y_3)] \)

\[
= (\nu_A \times \nu_B)[(x_1^* \times x_3^*), (y_1^* \times y_3^*)]
\]

\[
= (\nu_A(x_1) \lor \nu_B(y_1) \lor (\nu_A(x_3) \lor \nu_B(y_3))
\]

\[
= (\nu_A(x_1) \lor \nu_B(y_1) \land (\nu_A(x_3) \lor \nu_B(y_3))
\]

Now we have proving \( A \times B \) is an intuitionistic L-fuzzy H-ideal of \( X \times Y \).
And it can be extended for any \( n \) BF-algebras.

**Lemma 3.4**. Let \( A \) and \( B \) be two intuitionistic L-fuzzy subsets of \( X \) and \( Y \). If \( A \times B \) is an intuitionistic L-fuzzy H-ideal of \( X \) \( \times Y \) then following are true.

(i) \( \mu_A(x) \geq \mu_B(y) \) and \( \mu_B(y) \geq \mu_A(x) \) for all \( x \in X \), \( y \in Y \).

(ii) \( \nu_A(x) \leq \nu_B(y) \) and \( \nu_B(y) \leq \nu_A(x) \) for all \( x \in X \), \( y \in Y \).

**Proof**. Assume \( \mu_B(y) > \mu_A(x) \) and \( \mu_A(x) > \mu_B(y) \) for some \( x \in X \), \( y \in Y \).

Then \( (\mu_A \times \mu_B)(x, y) = \mu_A(x) \land \mu_B(y) \geq \mu_B(y) \land \mu_A(x) = (\mu_A \times \mu_B)(0, 0) \) which is a contradiction.

Similarly, assume \( \nu_A(x) < \nu_B(y) \) and \( \nu_B(y) < \nu_A(x) \) for some \( x \in X \), \( y \in Y \).
Then \( \{V(x,y)\} = V_A(x,y) \vee V_B(y) \leq V_B(y) \vee (V_A(x) \wedge V_B(y)) \) which is also a contradiction, thus proving the result.

**Theorem 3.5.** Let A and B be any two Intuitionistic L-fuzzy subsets of X and Y such that \( A \times B \) is an intuitionistic L-fuzzy H-ideal of \( X \times Y \). Then either A is an intuitionistic L-fuzzy H-ideal of X or B is an intuitionistic L-fuzzy H-ideal of Y.

Proof. Now by lemma 3.4 if we take \( \mu_A(0) \geq \mu_B(0) \) and \( \nu_A(0) \leq \nu_B(0) \) then
\[
(\mu_A \times \mu_B)(0,0) = (\mu_A(0) \wedge \mu_B(0)) = \mu_B(0)
\]

and
\[
(\nu_A \times \nu_B)(0,0) = \nu_B(0) \vee \nu_B(0) = \nu_B(y)
\]

(1)

Since \( A \times B \) is an intuitionistic L-fuzzy H-ideal of \( X \times Y \).

\[
(\mu_A \times \mu_B)[(x_1, y_1) \wedge (x_2, y_2)] \quad \wedge (\mu_A \times \mu_B)[x_2, y_2]
\]

(2)

Putting \( x_1 = x_2 = x_3 = 0 \) in (2) we get,
\[
(\mu_A \times \mu_B)((0,0), (0,0)]) \geq \nu_B(0,0) \wedge \nu_B(0,0)
\]

(3)

Using equation (1) in (3) we have
\[
\mu_B(0,0,0) \geq \mu_B(0,0,0) \wedge \mu_B(0,0)
\]

In the similar way we can prove
\[
\nu_B(0,0,0) \leq \nu_B(0,0,0) \vee \nu_B(0,0)
\]

This proves that B is an Intuitionistic L-fuzzy H-ideal of Y.

**Theorem 3.6.** Let A and B be any two Intuitionistic L-fuzzy H-ideals X and Y. Then \( A \times B \) is an Intuitionistic L-fuzzy H-ideal of \( X \times Y \) if and only if \( \nu_A \times \nu_B(x,y) \leq \nu_A(x) \vee \nu_B(y) \) and \( \nu_A \times \nu_B(x,y) \geq \nu_A(x) \vee \nu_B(y) \).

Proof. Let \( A \times B \) be an Intuitionistic L-fuzzy H-ideal of \( X \times Y \).

Clearly \( \mu_A \times \mu_B(x,y) = \mu_A(x) \wedge \mu_B(y) \) is L-fuzzy H-ideal of \( X \times Y \).

We have
\[
\nu_A \times \nu_B(x,y) = \nu_A(x) \vee \nu_B(y)
\]

\[1 - (\nu_A \times \nu_B(x,y)) = (1 - \nu_A(x)) \vee (1 - \nu_B(y))
\]

\[1 - (1 - \nu_A(x)) \vee (1 - \nu_B(y)) = \nu_A(x) \vee \nu_B(y)
\]

Thus \( \nu_A \times \nu_B(x,y) = \nu_A(x) \wedge \nu_B(y) \) is L-fuzzy H-ideal of \( X \times Y \).

Conversely, assume \( \nu_A \times \nu_B(x,y) \) and \( \nu_A \times \nu_B(x,y) \) are L-fuzzy H-ideal of \( X \times Y \).

Now
\[A \times B = \{X \times Y, \mu_A \times \mu_B, \nu_A \times \nu_B\}
\]

Since
\[
\nu_A \times \nu_B(x,y) = \nu_A(x) \vee \nu_B(y)
\]

we can easily observe that \( A \times B \) is an Intuitionistic L-fuzzy H-ideal of \( X \times Y \).

**Theorem 3.7.** Let A and B be any ILFS of X and Y. A and B are Intuitionistic L-fuzzy H-ideals of X and Y if and only if
\[
\Box(A \times B) = \{X \times Y, \mu_A \times \mu_B, \mu_A \times \mu_B\}
\]

\[\Diamond(A \times B) = \{X \times Y, \nu_A \times \nu_B, \nu_A \times \nu_B\}
\]

are Intuitionistic L-fuzzy H-ideals of \( X \times Y \).

Proof. Since
\[
(\mu_A \times \mu_B)(x,y) = \mu_A(x) \wedge \mu_B(y)
\]

and
\[
(\nu_A \times \nu_B)(x,y) = \nu_A(x) \vee \nu_B(y)
\]

the proof is clear.

**Theorem 3.8.** Let A and B be any two Intuitionistic L-fuzzy Closed H-ideals of X and Y. Then \( A \times B \) is an Intuitionistic L-fuzzy Closed H-ideal of \( X \times Y \).

Proof. Take \( (x,y) \in X \times Y \).

Then
\[
(\mu_A \times \mu_B)((0,0), (x,y)) = (\mu_A \times \mu_B)(0,0) \times (x,y)
\]

\[= \mu_A(0,0) \times \mu_B(x,y)
\]

\[\geq \mu_A(x) \times \mu_B(y)
\]

\[= (\mu_A \times \mu_B)(x,y)
\]

\[\forall x \in X \text{ and } y \in Y
\]

Now
\[
(\nu_A \times \nu_B)((0,0), (x,y)) = (\nu_A \times \nu_B)(0,0) \times (x,y)
\]

\[= \nu_A(0,0) \times \nu_B(x,y)
\]

\[\leq \nu_A(x) \times \nu_B(y)
\]

\[\forall x \in X \text{ and } y \in Y
\]

Thus \( A \times B \) is an Intuitionistic L-fuzzy Closed H-ideal of \( X \times Y \).

And it can be extended for any n BF-algebras.

One can easily prove theorem 3.6 and 3.7 for Intuitionistic L-fuzzy Closed H-ideals.

**Conclusion**

In this paper, we have extended the notion of the Cartesian product of Intuitionistic L-fuzzy sets to the notion of the generalized Cartesian product of Intuitionistic L-fuzzy H-ideals of BF-algebras. In [4] we have discussed the concept of Intuitionistic L-fuzzy subalgebras of BG-algebras. We expect that all the results proved in this paper can be proved for BG-algebras.
References
8. P.Muralikrishna and M.Chandramouleeswaran, A Note on Intuitionistic L-Fuzzy H-ideals of BF-Algebras, (Accepted in International Review of Fuzzy Mathematics)