Independent, perfect and connected neighborhood number of an M-strong fuzzy graph

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ABSTRACT
A neighborhood set $S \subseteq V$ of an M-strong fuzzy graph $G$ is said to be independent neighborhood set if $S$ is independent. $S$ is said to be perfect neighborhood set if all $u, v \in S, u \neq v$, the full fuzzy sub graphs $(N[u])$ and $(N[v])$ are edge disjoint. Also $S$ is said to be connected neighborhood set if full fuzzy sub graph $(S)$ is connected. The minimum scalar cardinality taken over all independent neighborhood set (perfect neighborhood set and connected neighborhood set) is called independent neighborhood number (perfect neighborhood number and connected neighborhood number). In this paper, these numbers are determined for various known fuzzy graphs and its relationship with some other known parameters of $G$ is investigated.
Definition: Two vertices \( u, v \in V \) of a fuzzy graph are said to be fuzzy independent if \( \mu(u, v) < \sigma(u) \land \sigma(v) \). A set \( S \subseteq V \) is said to be fuzzy independent set of \( G \) if any two vertices of \( S \) is independent.

Definition: A fuzzy graph \( G = (\sigma, \mu) \) is said to be complete fuzzy graph if \( \mu(u, v) = \sigma(u) \land \sigma(v) \) for all \( u, v \in V \).

The complement \( \tilde{G} = (\tilde{\sigma}, \tilde{\mu}) \) of a fuzzy graph \( G = (\sigma, \mu) \) is a fuzzy graph \( \tilde{G} = (\tilde{\sigma}, \tilde{\mu}) \), where \( \tilde{\sigma}(u) = \sigma(u) \) for all \( u \in V \) and \( \tilde{\mu}(u, v) = \sigma(u) \land \sigma(v) - \mu(u, v) \) for all \( u, v \in V \).

Definition: A fuzzy graph \( G = (\sigma, \mu) \) is said to be bipartite if the vertex set \( V \) can be partitioned into two non-empty sets \( V_1 \) and \( V_2 \) such that \( \mu(u, v) < \sigma(u) \land \sigma(v) \) if \( u \in V_1 \) and \( v \in V_2 \) or if \( u \in V_2 \) and \( v \in V_1 \). Then \( G \) is called fuzzy bipartite fuzzy graph.

Definition: A vertex \( u \) of a fuzzy graph \( G = (\sigma, \mu) \) is said to be isolated vertex if \( \mu(u, v) < \sigma(u) \land \sigma(v) \) for all \( v \in V \setminus \{u\} \).

Definition: A vertex \( u \) of a fuzzy graph \( G = (\sigma, \mu) \) is said to be connected if every pair of vertices has at least one fuzzy path between them, otherwise it is disconnected.

Definition: A cut vertex of a fuzzy graph \( G \) is one which whose removal disconnect the fuzzy graph.

Definition: A vertex \( u \) of a fuzzy graph \( G = (\sigma, \mu) \) is said to be isolated vertex if \( \mu(u, v) < \sigma(u) \land \sigma(v) \) for all \( v \in V \setminus \{u\} \). An edge \( e = (u, v) \) of a fuzzy graph is called an effective edge if \( \mu(u, v) = \sigma(u) \land \sigma(v) \). Here the vertex \( u \) is adjacent to \( v \) and the edge \( e \) is incident to \( u \) and \( v \). A fuzzy graph \( G = (\sigma, \mu) \) is said to be \( M \)-strong fuzzy graph if \( \mu(u, v) = \sigma(u) \land \sigma(v) \) for all \( (u, v) \in E \).

Definition: A set \( S \subseteq V \) is a neighborhood set \((n\)-set\) of an \( M \)-strong fuzzy graph \( G = (\sigma, \mu) \) if \( G = \bigcup_{u \in S} (\{N[u]\}) \), where \( \{N[u]\} \) is a full induced fuzzy sub graph of \( G \).

The neighborhood number of an \( M \)-strong fuzzy graph \( G \) is the minimum scalar cardinality taken over all \( n \)-set of \( G \) and is denoted by \( n_0 \).

Independent and Perfect neighborhood number

Definition: A neighborhood set \( S \subseteq V \) of an \( M \)-strong fuzzy graph \( G \) is said to be an independent neighborhood set if \( S \) is independent and is denoted by \( in\)-set. The minimum scalar cardinality taken over all \( in\)-set is called independent neighborhood number of \( G \) and is denoted by \( n_i \).

Definition: A neighborhood set \( S \subseteq V \) of an \( M \)-strong fuzzy graph \( G \) is said to be a perfect neighborhood set \((pn\)-set\) if all \( \alpha, \gamma \in S, \alpha \neq \gamma \), the full induced fuzzy sub graphs \( \{\{N[\alpha]\}\} \) and \( \{\{N[\gamma]\}\} \) are edge disjoint and is denoted by \( pn\)-set. The minimum scalar cardinality taken over all \( pn\)-set is called perfect neighborhood number of \( G \) and is denoted by \( n_{p} \).

Throughout this paper \( M \)-strong fuzzy graph \( G = (\sigma, \mu) \) alone are considered. \( M \)-strong fuzzy graph \( G = (\sigma, \mu) \) is simply denoted by \( G \) in the following sections.

Remarks:

1) There are fuzzy graph which has neither an \( in\)-set nor a \( pn\)-set. For example, any fuzzy cycle of odd length \( n \geq 5 \) has neither an \( in\)-set nor a \( pn\)-set.

2) Every \( pn\)-set is an \( in\)-set but converse is not true. Example: For example, from the fuzzy graphs \( G_1 \) and \( G_2 \) given in Fig.3.1, the neighborhood set \( \{u, v\} \) of \( G_1 \) is a \( pn\)-set and \( G_2 \) is an \( in\)-set whereas the neighborhood set \( \{u, v\} \) of \( G_2 \) is an \( in\)-set but not a \( pn\)-set.

Definition: A fuzzy graph \( G \) is an independent neighborhood fuzzy graph \((inf\)-graph\) if \( G \) has an \( in\)-set. A fuzzy graph \( G \) is a perfect neighborhood fuzzy graph \((pnf\)-graph\) if \( G \) has a \( pn\)-set.

Remark: Every \( pnf\)-graph is an \( inf\)-graph but not conversely. For example, from the Fig.3.1 \( G_1 \) is a \( pnf\)-graph and also \( inf\)-graph. \( G_2 \) is an \( inf\)-graph but not a \( pnf\)-graph.

Observation: In a fuzzy graph \( G \),

\[ n_0 = n_i = n_p = \min \{\sigma[1], \sigma[2]\} \] if \( G \) is a \( K_p \).

Observation: In a fuzzy graph \( G \),

\[ n_0 = n_i = n_p = \min \{\sigma[1], \sigma[2]\} \] if \( G \) is a \( K_{p, n_0} \).

Theorem: A fuzzy graph \( G \) is an \( inf\)-graph if and only if there exists a neighborhood set \( S \) such that \( V-S \) is a vertex cover.

Proof: For a neighborhood set \( S \) of a fuzzy graph \( G \), the set \( V-S \) is a vertex cover if and only if \( S \) is independent.

Theorem: If \( G \) is a \( pnf\)-graph of order \( p \), then \( 0 < n_0 < n_i < n_p < \beta_0 < p \).

Proof: Every vertex independent set of a perfect neighborhood set, every perfect neighborhood set is an independent neighborhood set and every independent neighborhood set is a vertex neighborhood set.

Hence, \( 0 < n_0 < n_i < n_p < \beta_0 < p \).
Example: Consider the following pnf-graph.

![Graph](image)

Here, \( n_0 = 0.1 \leq n_1 = 0.2 \leq \ldots \leq n_p = 2 \). 

**Theorem:** Given real numbers \( a, b, c, d, p \) with \( 0 \leq a \leq b \leq c \leq d \leq p \) and \( p \leq b + c + 1 \) then there exists a fuzzy graph \( G \) of order \( p \) such that \( n_0 = a, n_1 = b, n_p = c \) and \( \beta_0 = d \).

**Proof:** Consider a fuzzy path \( G_2 = (u_2, u_2) \) with 2 vertices \( u_1, u_2, u_3, \ldots, u_2 \) as given in the [Fig. 3.3].

![Graph](image)

Let \( \alpha = \sigma(u_2) = \sigma(u_2) \).

Let \( S = \{u_2, u_3, u_4, \ldots, u_2\} \) be a minimum neighborhood set with scalar cardinality \( a \). Hence \( n_0(G_2) = a \), where \( a \in \mathbb{R} \).

In \( G_1 \), replace the vertex \( u_1 \) by a complete fuzzy graph with 4 vertices and delete an edge from it and add an effective edge with pendent vertex \( u \) at \( u_2, u_3 \) such that \( \sigma(u) = \sigma(u_2) \) and \( \sigma(u) = \sigma(u_3) \). Then the fuzzy graph \( G_2 = (u_2, \mu_2) \) is obtained (as in Fig. 3.4).

![Graph](image)

Now, \( S \) is also a minimum \( n \)-set of \( G_2 \). Suppose \( b = \sigma(u_2) \) and \( c - b = \tau_2 \).

Add pendent effective edges, with scalar cardinality of the pendent vertices as \( \tau_2 \), at each of the vertices \( u_2, u_3 \) and \( u_2, u_3 \). Also, add pendent effective edges with scalar cardinality of the pendent vertices as \( \tau_2 - \sigma(u_2) \) at \( u_2 \). Then the fuzzy graph \( G_3 = (u_2, \mu_3) \) is obtained (as in Fig. 3.5).

![Graph](image)

Now, we show that \( G_3 \) is the required fuzzy graph. Let \( S_1, S_2, S_3 \) be the set of all pendent vertices adjacent to \( u_2, u_2, u_3 \) and \( u_2, u_3 \) respectively. Then \( |S_1| = \tau_2, |S_2| = \tau_1 + \sigma(u_2), |S_3| = \tau_1 + \sigma(u_2) \).

Clearly, the set \( S \) is a minimum \( n \)-set of \( G_3 \) and \( n(S_0) = a \).

Also \( \{u_2, u_2, u_3, \ldots, u_2\} \cup S_2 \) is a minimum in-set with scalar cardinality is \( a = \sigma(u_2) + \tau_1 + \sigma(u_2) = a = b \).

Further the set \( S_1 \cup \{v\} = \{u_2, u_3, \ldots, u_2\} \cup S_2 \cup S_3 \) is a minimum in-set with scalar cardinality is \( \tau_2 + \sigma(v) + \sigma(u_2) - \sigma(u_2) - \sigma(u_2) + \sigma(u_2) \leq \tau_2 + \sigma(u_2) + \sigma(u_2) \leq b + c + 1 \).

The order of \( G_3 \) is the scalar cardinality of \( S_1 \cup \{u_2, v, w\} \cup \{u_2, u_3, u_4, \ldots, u_2\} \cup S_2 \cup S_3 \).

\( \mu = \tau_2 + \sigma(v) + \sigma(u_2) + 2\sigma(u_2) - \sigma(u_2) - \sigma(u_2) + \sigma(u_2) + \sigma(u_2) \leq \tau_2 + \sigma(u_2) + \sigma(u_2) \leq b + c + 1 \).

Connected neighborhood set

**Definition:** A neighborhood set \( S \subseteq V \) of a fuzzy graph \( G \) is a connected neighborhood set if full induced fuzzy sub graph \( \langle S \rangle \) is connected and is denoted by \( cn \). The minimum scalar cardinality taken over all \( cn \)-set is called connected neighborhood number of \( G \) and is denoted by \( n_c \).

**Example:** Consider the fuzzy graph in Fig. 4.1.

![Graph](image)

Therefore \( n_2 = n_0 \).
Remarks:
1) Every connected neighborhood set is a neighborhood set, i.e. \( n_G(v) \leq n_G(v) \).
2) Every independent neighborhood set is a neighborhood set, i.e. \( i_G(v) \leq n_G(v) \).
3) Every connected neighborhood set is a connected dominating set, i.e. \( c_G(v) \leq n_G(v) \).
4) There is no relation between \( n_G(v) \) and \( n_G(v) \). For example, consider the fuzzy graph in Fig. 4.2.

![Fig. 4.2: \( G_2 \)](image)

\[ n_i \text{-set} = \{ v_1, v_3 \} \implies n_i = 0.8 \text{ and } n_r \text{-set} = \{ v_2, v_3 \} \implies n_r = 0.5. \]
Therefore \( n_i \leq n_r \). In Example 4.2, we obtain that \( n_i \leq n_r \).

Hence no relation exist between \( n_i \) and \( n_r \).

Observation: In a complete fuzzy graph \( K_2 \),

\[ n_c = \min_{u \in V} \sigma(u). \]

Observation: In a fuzzy cycle \( C_n \) of order \( p \) with \( n \) vertices, \( n \geq 4 \),

\[ n_c = p - \min_{u \in V} \sigma(u). \]

Theorem: In a complete bipartite fuzzy graph \( K_{m,n} \),

\[ |\sigma(a)| \leq |\sigma(b)|, \]

\[ n_c = \begin{cases} |\sigma(a)| & \text{if } |\sigma(a)| = 1 \\ |\sigma(a)| & \text{if } |\sigma(a)| = 1 \text{ and } |\sigma(b)| \neq 1 \\ |v_1| + \min_{u \in V} \sigma(u), & \text{otherwise} \end{cases} \]

Proof: Case (i): If \( |\sigma(a)| = 1 \), then every vertex of \( V_2 \) is a neighborhood of \( u \in V_1 \). Hence \( n_c = \sigma(a) = |\sigma(a)| \).

Case (ii): If \( |\sigma(a)| = 1 \text{ and } |\sigma(b)| \neq 1 \), then every vertex of \( V_1 \) is a neighborhood of \( v \in V_2 \). Hence \( n_c = \sigma(a) = |\sigma(b)| \).

Case (iii): Let \( K_{m,n} \) be the complete bipartite fuzzy graph, defined on \( V_1 \) and \( V_2 \) with \( |\sigma(a)| \leq |\sigma(b)| \) and \( v \in V_2 \) such that \( \sigma(v) \) is a minimum weight. Since \( V_1 \) is a neighborhood set, since \( K_{m,n} \) is a complete bipartite and \( V_1 \cup \{ v \} \) is also a neighborhood set and its full fuzzy induced sub graph is connected. Hence, \( n_c = |\sigma(a)| + \min_{u \in V} \sigma(u) \).

Observation: For a fuzzy tree \( T \) of order \( p \),

\[ n_c = p - t, \]

where \( t \) is the scalar cardinality of the set of all the pendant vertices in \( T \).

Note: Let \( |E| \) denote the maximum scalar cardinality of the set of all pendant vertices in any full spanning fuzzy tree of \( G \).

Theorem: In any connected fuzzy graph \( G \) of order \( p \), any cut vertex of \( G \) belongs to each connected neighborhood set and \( k \leq n_c \leq p - \frac{|E|}{2} \), where \( k \) is the minimum scalar cardinality of cut vertices of \( G \).

Proof: Let \( S \) be a connected neighborhood set, and \( v \) be a cut vertex of \( G \). Suppose \( v \in S \), clearly \( v \) lies on two blocks \( B_1 \) and \( B_2 \) (say). There exists a vertex \( u \in B_1 \) and \( w \in B_2 \) such that \( v \subseteq S \) and both of them different from \( v \). Also \( u-w \)

fuzzy path contains \( v \) since \( v \subseteq S \), the full fuzzy induced sub graph \( (S) \) is disconnected, which is a contradiction. Hence \( v \subseteq S \) and \( k \leq n_c \).

Now, consider a full spanning fuzzy tree \( T \) of \( G \) with maximum scalar cardinality of pendant vertices. Let \( W \) be the set of all non-pendant vertices of \( T \). Delete all the vertices of \( G \) which belongs to \( W \). Let \( G' \) be the resultant graph with \( W' = V - W \) vertices and \( E' \) edges. Let \( S \) be the minimum neighborhood set of \( G' \) after deleting the isolated vertices in \( G \).

Then \( |S| \leq \frac{|E|}{2} \) and clearly \( W' \cup S \) is a connected neighborhood set of \( G \). Hence, \( n_c \leq p - |E| + \frac{|E|}{2} = p - \frac{|E|}{2} \).

Corollary: If \( G \) is complete fuzzy graph, then \( G' \) is also complete. Hence, \( n_c = |W'| = \min_{u \in V} \sigma(u) \).

Corollary: If \( G \) is tree, then \( G' \) is a null graph. Hence \( n_c = |W'| = p - |E| \).

Example: Consider the fuzzy graph

![Graph](image)

Bold lines indicates the spanning tree \( T \) of \( G \).  \( v_1 \) and \( v_2 \) are the non-pendent vertices of \( T \) and \( \{ v_3, v_4 \} \) is a minimum \( n \)-set of \( G' \). Therefore \( \{ v_1, v_2, v_3, v_4 \} \) is a \( n \)-set. The \( n_i \)-set is \( \{ v_1, v_2, v_3 \} \). Hence \( n_i \leq p - \frac{|E|}{2} \implies 1.1 \leq 2.3 \).

Observation: Let \( G \) be a connected fuzzy graph of order \( p \) with not less than two vertices which is neither complete nor a fuzzy cycle. Then \( G \) has at least one pair of non adjacent vertex \( u,v \) such that the fuzzy graph \( G \{u,v\} \) is connected neighborhood set.

Theorem: Let \( G \) be a connected fuzzy graph of order \( p \). Then \( n_c = p - \max_{u \in V} \sigma(u) \) if and only if \( G \) is complete fuzzy graph with 2 vertices or \( G \) is a fuzzy cycle with four or more vertices.

Proof: Clearly, \( n_c = p - \max_{u \in V} \sigma(u) \) if \( G \) is complete fuzzy graph with two vertices or \( G \) is a fuzzy cycle with four or more vertices.

Conversely, suppose \( n_c = p - \max_{u \in V} \sigma(u) \) and \( G \) is not a complete fuzzy graph with vertices 2. Then \( G \neq K_2 \) with vertices more than 3, for otherwise \( n_c = \min_{u \in V} \sigma(u) < p - \max_{u \in V} \sigma(u) \) which is a contradiction. If \( G \) is not a fuzzy cycle, then the observation...
there exists two non-adjacent vertices \( u, v \) in \( G \) such that at least one of them have maximum weight. Then the fuzzy graph \( G' = G - \{u, v\} \) is connected. Clearly \( V(G') \) is a connected neighborhood set of \( G \) of scalar cardinality \( \nu - \sigma(u) - \sigma(v) \). Therefore \( n_2 \leq \nu - \sigma(u) - \sigma(v) \).

**4.15 Corollary:** Let \( G \) be a connected fuzzy graph of order \( p \) and size \( q \) except complete fuzzy graph with 2 vertices. If \( G \) is not a fuzzy cycle, then

\[
\frac{\epsilon}{2} \leq p - (\sigma_1 + \sigma_2)
\]

Where \( \sigma_1 \) and \( \sigma_2 \) are the successive minimum weights of the vertices in \( G \).

**References**


