Self-adjointness of product of two high-order singular differential operators
Qiuxia Yang1 and Wanyi Wang2

1Department of Computer Science and Technology, Dezhou University, Dezhou 253023, P.R. China
2Mathematics Science College, Inner Mongolia University, Huhhot 010021, P.R. China.

ABSTRACT
The self-adjointness of product of two singular differential operators generated by 2th-order symmetric differential expression is discussed. By means of the construction theory of real parameter on differential operators and matrix computation, the necessary and sufficient conditions which make the product being the self-adjoint operators in are obtained.

Preliminaries
We consider the formal symmetric ordinary differential expression of order 2n on the interval (a, b), given by
\[ l(y) = \sum_{j=0}^{\infty} y^{(j)}(p_j y^{(j)})_0, \quad x \in (a, b), \quad -\infty < a < b \leq \infty. \] (1)

We assume that the real coefficients \( p_j(t) \) satisfy the following basic conditions:
\[ p_0(x) > 0 \quad \text{and} \quad p_{n-j}(x) \in C^{2r+j}(a,b), \quad j = 0, 1, 2, L, n. \] (2)

The basic conditions ensure that the formal square \( l^2 \) of \( l \), defined by
\[ l^2 y = l(ly) \] (3)
exists as a differential expression. Furthermore, \( l \) and \( l^2 \) are regular on \([a, b]\) and singular on endpoint \( b \).

Note that in this paper we will write a matrix \( A \) with \( m \) rows and \( n \) columns as \( A = (a_{ij})_{m \times n} \), where \( a_{ij} \) is the element of \( A \) appearing in the \( i \) th row and \( j \) th column. Let \( A^T \) and \( A^* \) denote the transpose and complex conjugation transpose of \( A \) respectively.

In the following, one can easily see that the facts and notations we introduce for \( l \) are applicable for \( l^2 \) similarly.

Let \( n_+ \) and \( n_- \) denote the deficiency indices of the formally symmetric differential expression \( l \) on \([a, b]\) associated with the upper and lower half-planes respectively.

Write \( \text{def}(l) = (n_+, n_-) \). Here, \( n_+ \) and \( n_- \) are not necessarily equal.

Definition 1 [11] Under the basic conditions (2), we say that \( l^2 \) is partially separated in \( L^2((a, b)) \) if \( y \in L^2((a, b)) \), \( y^{(4n-4)} \in AC_{loc}((a, b)) \) and \( l^2(y) \in L^2((a, b)) \) together imply that \( l(y) \in L^2((a, b)) \).

Introduction
Let \( l \) be a formal symmetric ordinary differential expression of order \( 2n \) on the interval \((a, b)\). We assume that the product \( l^2 \) can be well formed. Let \( L \) be a differential operator generated by \( l \) in \( L^2((a, b)) \) with some boundary conditions (11).

It is known that the product \( l^2 \) is a formally symmetric differential expression of order \( 4n \) on \((a, b)\) with equal deficiency indices, and the minimal operator associated with \( l^2 \) in \( L^2((a, b)) \) is a positive symmetric differential operator (cf. [2, 5]).

There have been many research results on the subject of the products or powers of differential operators, especially on deficiency indices of powers of \( l \) and commutativity of differential expressions (see [2-4]).

On the problem of determining the self-adjointness of the product of two differential operators, it has been considered in [6] when \( l \) is a limit-circle Sturm-Liouville operator on \((a, b)\).

By means of the construction theory of differential operator and matrix computation, a necessary and sufficient condition for the self-adjointness of the product two high-order differential expressions (see [2-4]) on the interval \( 2 \times 2 \), where \( a_{ij} \) is the element of \( A \) appearing in the \( i \) th row and \( j \) th column. Let \( A^T \) and \( A^* \) denote the transpose and complex conjugation transpose of \( A \) respectively.

In the following, one can easily see that the facts and notations we introduce for \( l \) are applicable for \( l^2 \) similarly.

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Definition 1 [11] Under the basic conditions (2), we say that \( l^2 \) is partially separated in \( L^2((a, b)) \) if \( y \in L^2((a, b)) \), \( y^{(4n-4)} \in AC_{loc}((a, b)) \) and \( l^2(y) \in L^2((a, b)) \) together imply that \( l(y) \in L^2((a, b)) \).
Lemma 1 Let (2) hold. Then \( l^2 \) is partially separated in \( L^2((a,b)) \) if and only if
\[
\text{def} \ (l) = (n_+ + n_- n_+ + n_-). \tag{4}
\]

Proof
We define the operator \( L_M(l) \) by \( L_M(l)y = ly \),
\[y \in D_M(l), \]
where \( D_M(l) = \{(y(\frac{t}{2})) \mid (2) \} \in AC(\frac{a,b})I(\theta(\frac{t}{2})) \}

The operator \( L_M(l) \) is called the maximal operator of \( l \) on \([a,b]\) and its domain \( D_M(l) \) is defined as the maximal domain of \( l \) on \([a,b]\). It is known that \( D_M(l) \) is dense in \( L^2((a,b)) \).

Thus the adjoint \( L_M(l) \) is well defined. Let \( L_0(l) = L_M(l) \), the operator \( L_0(l) \) is called the minimal operator of \( l \) on \([a,b]\) and \( D_0(l) \) is defined as the minimal domain of \( l \) on \([a,b]\).

For any \( y, z \in D_M(l) \), we have the Green's formula,
\[
\int_a^b (l(y)z - y(l(z)))dt = \{y, z\}_{2n}(b) - \{y, z\}_{2n}(a), \tag{5}\]
\[\text{where} \ [y, z]_{2n}(x) = R(z)Q_{2n}(x)C(y), \]
\[C(y) = \begin{bmatrix} y(x) \\ M \end{bmatrix}, \quad R(z) = \begin{bmatrix} (z(x), L) \end{bmatrix}(x), \quad Q_{2n}(x) = \begin{bmatrix} [q_{jk}] \mid (j, k = 1,2, L, 2n) \end{bmatrix}, \quad [\cdot, \cdot]_{2n} \text{ is called the Lagrange bilinear form corresponding to } (l) \text{ on } [a,b].\]

Lemma 2 \([9]\) \( y \in D_0(l) \) holds if and only if
(i) \( y(a) = y'(a) = L = y^{(2n)}(a) = 0 \),
(ii) \( \forall z \in D_0(l), \quad [y, z]_{2n}(b) = 0 \).

Let \( z_j (j = 1,2, L, 2n) \) be a set of functions in \( D_0(l) \) which satisfy the following conditions:
\[
z_j^{(k)}(a) = \delta_{jk}, \quad z_j^{(k)}(a) = 0, \quad z_j(x) = 0, \quad a < a < x < b, (j, k = 1,2, L, 2n). \tag{6}\]

Lemma 3 \([12]\) If \( \text{def}(l) = (m, m) \), then for any \( \lambda_0 \in \mathbb{R} \), the number of solutions of \( \lambda_0 \) is not more than \( m \).

Lemma 4 \([10]\) Let \( \lambda_0 \in (\mu_1, \mu_2) \) and \( \theta_1, \theta_2, L, \theta_m \in L^2((a,b)) \) be linearly independent square integrable solutions of \( l y = \lambda_0 y \). Then there must exist \( \theta_1, \theta_2, L, \theta_{2m-2n} \), satisfy the following conditions:
\[
\text{rank}([\theta_1, \theta_2]_{12})_{15(i, j=2m-2n)} = 2m - 2n \quad \text{and} \quad D_0(l) = D_M(l) \quad \text{such that} \quad [\theta_1, \theta_2, L, \theta_{2m-2n}] \]
\[
\text{where the symbol } \& \text{ denotes a direct sum,} \quad \text{span} \{z_1, z_2, L, z_{2n}\} \text{ denotes the linear span of } z_1, z_2, L, z_{2n} \text{ and} \quad \text{span} \{\theta_1, \theta_2, L, \theta_{2m-2n}\} \text{ denotes the linear span of} \quad \{\theta_1, \theta_2, L, \theta_{2m-2n}\}. \]

Definition 2 Under the assumption that Lemma 3, \( \theta_1, \theta_2, L, \theta_{2m-2n} \) are called the second \( L^2 - \) solutions of \( l y = \lambda_0 y \).

Let
\[
B = ([\theta_1, \theta_2])_{15(i, j=2m-2n)}^T \]
\[\text{where} \ \theta_1, \theta_2, L, \theta_{2m-2n} \text{ are the second } L^2 - \text{ solutions of} \quad l y = \lambda_0 y. \quad \text{Then we have the following lemma:} \]

Lemma 5 \([10]\) Let \( l y \) be a closed symmetric operator with deficiency indices \( (m, m) \). Then linear manifold \( D \subset D_M(l) \) is the self-adjoint extension domain of \( L_0 \) if and only if there exist numerical matrices \( M_{m \times 2n} \) and \( N_{m \times (2m-2n)} \), satisfy:
\[
(1) \quad \text{rank}(M \oplus N) = m; \\
(2) \quad MQ^{-1}(a)M^* + NBN^* = 0; \\
(3) \quad D = \{y \in D(L_0) : \begin{bmatrix} M & M \end{bmatrix} - N \begin{bmatrix} M & M \end{bmatrix} = 0 \}.
\]

Self-adjointness of product of two operators on \((a,b)\)
In this section, we always assume that the basic conditions (2) hold and \( l^2 \) is partially separated in \( L^2((a,b)) \). Let
\[
l y = y^{(2n)} + q(t)y, \tag{7}\]
\[\text{where} \quad n \in \mathbb{N}, t \in (a,b) \quad \text{and} \quad q(t) \text{ is real function. We easily see from (7) that} \]
\[
Q_{2n}(t) = \begin{bmatrix} 0 & 0 & L & 0 & -1 \\ 0 & 0 & L & 1 & 0 \\ M & M & O & M & M \\ 0 & -1 & L & 0 & 0 \\ 1 & 0 & L & 0 & 0 \end{bmatrix}. \tag{8}\]

Then \( Q_{2n}(a) = -I_{2n \times 2n} \), \( Q_{2n}(a) = -Q_{2n}(a) \).
Here assume that \( \text{def}(l) = (m, m) \). By Lemma 4, the second \( L^2 - \) solutions \( \theta_1, \theta_2, L, \theta_{2m-2n} \) of \( l y = 0 \) satisfy
\[
(C_n(\theta_1)C_n(\theta_2)(a)L\theta_{2m-2n}) = W(\theta(a)) = \begin{bmatrix} O_{2n \times (2m-2n)} \\ Q_{2n \times (2m-2n)} \end{bmatrix}, \]
\[\text{where} \quad C_{2n}(\theta_i)(a) = (\theta_k(a), \theta_k(a)L, \theta_{2m-2n}(a)) \text{ and} \quad W(\theta(t)) \text{ denotes the Wronskian matrix of} \quad \{\theta_i(t); i = 1,2,L,2m-2n}. \quad \text{Let} \quad B_{2m-2n} = ([\theta_1, \theta_2]_{2n})^T, \ \text{then} \]

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Proposition 6 If def \( l = (m, m) \), then def \( l^2 = (m', m'^* \) and \( 2m \leq m', m'^* \leq m + 2n \).

Proof If def \( l = (m, m) \), then \( \Psi_1, \Psi_2, L, \Psi_{2n-m} \) are the first \( L^2 \) solutions of \( ly = 0 \) and are also the first \( L^2 \) solutions of \( l^2(y) = 0 \). In fact, for

\[ B_{2m-2n}(b) = B_{2m-2n}(a) = ([\theta, \theta]_{2n}(a))^T = (W(\theta(a)))^T Q_{2n}(\mathbf{w}) \in \mathbf{G}(\partial \theta(a), \partial \theta(a)), \{\psi_1, \psi_2, y(b) = 0 \}. \quad D(l^2) \subset D(l) \subset D_M(l). \]

So \( l^2(y) \) have middle defect indices \( (m', m'^*) \). It follows from \( \Psi_1 = iy, L \Psi_2 = -iy, L^2(y) = -y \) has at least \( 2m \) linearly independent square integrable solutions on \( [a, b] \), i.e., \( 2m \leq m', m'^* \leq m + 2n \).

Corollary 7 If \( l(y) \) is limit-circle case, then \( l^2(y)(k = 1, 2, L) \) is also limit-circle case.

Lemma 8 If def \( l = (m, m) \), then def \( l^2 = (2m, 2m) \). Proof If def \( l = (m, m) \). Under the basic conditions (2), it is known that \( l^2 \) is a formally symmetric differential expression of order \( 4n \) on \( [a, b] \). Furthermore, it follows from (4) that def \( l^2 = (2m, 2m) \).

In the following, similar to the notations of Section 1, let \( L_0(l^2) \) and \( L_M(l^2) \) be the minimal and maximal operators generated by \( l^2 \) in \( \mathcal{L}((a, b)) \). \( D_0(l^2) \) and \( D_M(l^2) \) be the domains of \( L_0(l^2) \) and \( L_M(l^2) \). Let \( \varphi_1, \varphi_2, L, \varphi_{2m-2n} \) be the second \( L^2 \) solutions of \( l^2 \) and satisfy

\[ (C_{4n}(\varphi_1)(a), C_{4n}(\varphi_2)(a), L, C_{4n}(\varphi_{2n-2m})(a)) = \begin{bmatrix} 0_{(2n-m)(2n-2n)} \\ I_{(2n-2n)(2n-2n)} \\ (2n-2n)(2n-2n) \end{bmatrix} \]

From Lemma 4, \( \varphi_i \in \mathcal{L}((a, b)) (i = 1, 2, L, 2m - 2n) \). Furthermore \( l(\theta) = 0 \), \( l^2(\theta) = l(\theta) \), then \( \theta \), \( i = 1, 2, L, 2m - 2n \) is the solution of \( l^2 \). The matrix of \( \theta_1, \theta_2, L, \theta_{2m-2n}, \varphi_1, \varphi_2, L, \varphi_{2m-2n} \) is defined by

\[ (C_{4n}(\theta_1)(a), L, C_{4n}(\theta_{2m-2n})(a)) = \psi_1, 0_{(2n-2n)(2n-2n)} \]

where \( \psi_1 = \psi_1, 0_{(2n-2n)(2n-2n)} \). So

\[ \theta_1, \varphi_1 (i = 1, 2, L, 2m - 2n) \) are the second \( L^2 \) solutions of \( l^2(y) = 0 \) and

\[ B_{2m-2n}(b) = B_{2m-2n}(a) = ([\xi, \xi]_{2n}(a))^T = (W(\theta(a)))^T Q_{2n}(\mathbf{w}) \in \mathbf{G}(\partial \theta(a), \partial \theta(a)), \{\psi_1, \psi_2, y(b) = 0 \}. \quad D(l^2) \subset D(l) \subset D_M(l). \]

with \( [\xi, \xi]_{2n}(a) = \psi_1, \psi_1 \).
Thus for \( \phi \in D_M(l^2) \subset D_M(l) \), from Lemma 4, it is easily seen that

\[
\phi_i = y_{0i} + \sum_{s=1}^{2m-2n} d_{is} z_s + \sum_{j=1}^{2m-2n} a_{ij} \theta_j, \quad i = 1, 2, L, 2m-2n. \tag{16}
\]

where \( y_{0i} \in D_0(l), d_{is}, a_{ij} \) are real constants.

In addition, for \( \forall y \in D_0(l) \),

\[
y = y_0 + \sum_{i=1}^{2n} d_i z_i + \sum_{i=1}^{2n} \bar{c}_i \theta_i + c_i^* \phi_i,
\]

(17)

where \( y_0 \in D_0(l^2), \bar{d}_i, \bar{c}_i, c_i^* \) are also real constants. From Lemma 2, (8), (15) and (17), we obtain

\[
\begin{bmatrix}
[y, \theta]_{2n}(b) \\
[y, \theta]_{2n}^{2m-2n}(b) \\
[y, \phi]_{2n}^{2m-2n}(b)
\end{bmatrix} =
\begin{bmatrix}
c_1 \\
c_1^* \\
0
\end{bmatrix}
\begin{bmatrix}
0 \\
\bar{c}_1, c_i^* \\
\bar{c}_1, c_i^*
\end{bmatrix}
\begin{bmatrix}
M \\
M \\
M
\end{bmatrix}
\]

Thus

\[
\begin{bmatrix}
\bar{c}_1 \\
\bar{c}_{2m-2n} \\
c_i^* \\
c_i^*
\end{bmatrix}
= \begin{bmatrix}
0 \\
\bar{Q}_{2m-2n}^{-1}(a) \\
0 \\
0
\end{bmatrix}
\begin{bmatrix}
[y, \theta]_{2m-2n}^{4n}(b) \\
y, \phi]_{2m-2n}^{4n}(b)
\end{bmatrix}
\]

(18)

By (16), (17), \( y \) can be defined by

\[
y = y_0 + \sum_{i=1}^{2n} d_i z_i + \sum_{i=4}^{2n} \bar{c}_i \theta_i + \sum_{i=4}^{2n} \bar{c}_i \theta_i + \sum_{i=4}^{2n} \bar{c}_i \theta_i.
\]

From Lemma 2, (8), (9) and (18), we have

\[
\begin{bmatrix}
[y, \theta]_{2n}(b) \\
[y, \theta]_{2n}^{2m-2n}(b) \\
y, \phi]_{2n}^{2m-2n}(b)
\end{bmatrix} =
\begin{bmatrix}
M \\
M \\
M
\end{bmatrix}
\begin{bmatrix}
\bar{c}_1 \\
\bar{c}_{2m-2n} \\
c_i^* \\
c_i^*
\end{bmatrix}
\begin{bmatrix}
0 \\
\bar{Q}_{2m-2n}^{-1}(a) \\
0 \\
0
\end{bmatrix}
\begin{bmatrix}
[y, \theta]_{2m-2n}^{4n}(b) \\
y, \phi]_{2m-2n}^{4n}(b)
\end{bmatrix}
\]

(19)

By (7), we obtain

\[
y(y) = \begin{bmatrix}
y, \phi]_{2m-2n}^{4n}(b) \\
y, \phi]_{2m-2n}^{4n}(b)
\end{bmatrix} =
\begin{bmatrix}
M \\
M
\end{bmatrix}
\begin{bmatrix}
[y, \theta]_{2m-2n}^{4n}(b) \\
y, \phi]_{2m-2n}^{4n}(b)
\end{bmatrix}
\]

(19)
Theorem 11  

\[
\text{Rank}(M \oplus N) = 2m.  
\]

Proof  

Since \( M = \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix} \begin{pmatrix} I_{2n \times 2n} & 0 \\ 0 & I_{2n \times 2n} \end{pmatrix} \) is self-adjoint operator if and only if \( A^T = A \) and \( C^T = C \), we have  

\[
(M \oplus N) = \begin{pmatrix} M_1 & 0 \\ 0 & N_2 \end{pmatrix} = P Q.  
\]

And \( \det Q = \det M_2 \cdot \det N_2 = 1 \times (-1) \neq 0 \), i.e., \( Q \) is invertible. So \( \text{Rank} PQ = \text{Rank} P = 2m \), i.e., \( \text{Rank}(M \oplus N) = 2m \).

Theorem 12  

\( L = L_1 L_2 \) is self-adjoint operator if and only if \( AQ_2n(a)C^* + BB_{2m-2n}(b)D^* = 0 \).

Proof  

By (13), (21) and Lemma 10 (i), we have  

\[
MQ_{2n}^{-1}(a)M^* = \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix} \begin{pmatrix} 0 & Q_{2n}^{-1}(a) \\ Q_{2n}^{-1}(a) & 0 \end{pmatrix} A^* C^*  
\]

\[
= \begin{pmatrix} 0 & AQ_{2n}(a) \end{pmatrix} \begin{pmatrix} A & 0 \end{pmatrix} C = \begin{pmatrix} 0 & AQ_{2n}(a)A^* \end{pmatrix} C  
\]

Similarly, from (15), (19) and Lemma 10 (ii), we obtain  

\[
NB_{2m-2n}(b)N^* = \begin{pmatrix} 0 & BQ_{2m-2n}(a)D^* \end{pmatrix}  
\]

Thus, \( MQ_{2n}^{-1}(a)M^* + NB_{2m-2n}(b)N^* = 0 \) if and only if \( AQ_{2n}(a)C^* + BB_{2m-2n}(b)D^* = 0 \).

Theorem 13  

\( L = L_1 L_2 \) is self-adjoint operator if and only if \( AQ_{2n}(a)C^* - BQ_{2m-2n}(a)D^* = 0 \)

\[
(AQ_{2n}(a)C^* + BQ_{2m-2n}(a)D^* = 0).  
\]

If \( L_1 = L_2 \), then \( L = L_1^2 \) and \( A = C, B = D \). By Lemma 5 and Theorem 13, we can conclude the following conclusion:  

Corollary 14  

\( L = L_1^2 \) is self-adjoint operator if and only if \( L_1 \) is self-adjoint operator.

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