Effects of thermal radiation and chemical reaction on MHD convective flow of a polar fluid past a moving vertical plate with viscous dissipation

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ABSTRACT
The objective of the present investigation is to analyze the radiation and mass transfer effects on an unsteady two-dimensional laminar convective boundary layer flow of a viscous, incompressible, chemically reacting and dissipative fluid along a semi-infinite vertical plate with suction. The equations of continuity, linear momentum, energy and diffusion, which govern the flow field are solved by using a regular perturbation method. The behavior of the velocity, temperature, concentration has been discussed numerically and graphically for variations in the governing parameters.

Keywords
MHD, Thermal radiation, Chemical reaction, Viscous dissipation.

Introduction
The study of heat and mass transfer with chemical reaction is of great practical importance to engineers and scientists because of its almost universal occurrence in many branches of science and engineering. Das et al. [1] have studied the effects of mass transfer on the flow past an impulsively started infinite vertical plate with constant heat flux and chemical reaction. Diffusion of chemically reactive species from a stretching sheet is studied by Anderson et al. [2]. Anjalidevi and Kandaswamy [3, 4] have analysed the effects of chemical reaction, heat and mass transfer on laminar flow without or with along a semi-infinite horizontal plate. The effects of the chemical reaction and mass transfer on MHD unsteady free convection flow past a semi infinite vertical plate with constant/variable suction and heat sink was analyzed by [5-7]. Ghaly and Seddeek [8] have discussed the Chebyshev finite difference method for the effects of chemical reaction, heat and mass transfer on the laminar flow along a semi-infinite horizontal plate with temperature dependent viscosity.

A new stage in the evaluation of fluid dynamic theory is in progress because of the increasing in the processing industries and elsewhere of materials whose flow shear behavior cannot be characterized by Newton relationships. The theory of micropolar fluids was first introduced and formulated by Eringen [9]. This theory displays the effects of local rotary inertia and couple stress. The theory is expected to a mathematical model for the non-Newtonian fluid behavior observed in certain fluid such as exotic lubricants, colloid fluids, liquid crystals etc., which is more realistic and important from a technological point of view. The theory of thermomicrofluidic fluids was developed by Eringen [10] by extending his theory of micropolar fluid. The flow characteristics of the boundary layer of micropolar fluid over a semi-infinite plate in different situations have been studied by many authors in Refs. [11, 12]. We know that the radiation effect is important under many non-isothermal situations. If the entire system involving the polymer extrusion process is placed in a thermally controlled environment, then radiation could become important. The radiative flows of an electrically conducting fluid with high temperature in the presence of a magnetic field are encountered in electrical power generation, astrophysical flows, solar power technology, space vehicle re-entry, nuclear engineering applications and other industrial areas [13-16]. In the above authors studied the radiation on Newtonian and non-Newtonian fluids with and without magnetic field has been considered by many authors.

In all the studies mentioned above, viscous dissipation is neglected. But the viscous dissipation in the natural convective flow is important, when the flow field of extreme size or in high gravitational field. Gebhart [17] show the importance of viscous dissipative heat in free convection flow in the case of isothermal and constant heat flux at the plate. Gebhart and Mollendorf [18] considered the effects of viscous dissipation for the external natural convection flow over a surface. Soudalgekar [19] analyzed viscous dissipative heat on the two dimensional unsteady free convection flow past an infinite vertical porous plate when the temperature oscillates in time and there is constant suction at the plate.

However, the interaction of radiation with mass transfer in a chemically reacting and dissipative fluid has received little attention. So the objective of this paper is to study the effects of chemical reaction and thermal radiation on MHD convective flow of micro polar fluid past a semi infinite vertical plate with viscous dissipation.

Mathematical analysis:
Consider the steady, laminar, two-dimensional, free convection flow of a viscous incompressible, electrically conducting, and polar fluid occupying a semi-infinite region of the space bounded by an infinite vertical porous plate in the...
presence of viscous dissipation and subjected to thermal radiation. The $x^*$-axis is taken along the vertical plate in an upward and $y^*$-axis is taken normal to the plate. The transversely applied magnetic field and magnetic Reynolds number are assumed to be very small so that the induced magnetic field is negligible [20]. The governing equations for this physical situation are based on the usual balance laws of mass, laminar momentum, angular momentum, energy and mass diffusion modified to account for the physical effects mentioned above.

The equations are given by continuity:

$$\frac{\partial v^*}{\partial y^*} = 0,$$

(2.1)

linear momentum:

$$v^* \frac{\partial u^*}{\partial y^*} = (\gamma + \nu) \frac{\partial^2 w^*}{\partial y^*^2},$$

(2.2)

angular momentum:

$$v^* \frac{\partial \omega^*}{\partial y^*} = \frac{v^*}{T} \frac{\partial^2 \omega^*}{\partial y^*^2},$$

(2.3)

energy:

$$v^* \frac{\partial T^*}{\partial y^*} = \alpha \left( \frac{\partial^2 T^*}{\partial y^*^2} - \frac{1}{K^*} \frac{\partial q^*}{\partial y^*} \right) + \frac{\nu}{C_p^*} \left( \frac{\partial u^*}{\partial y^*} \right)^2,$$

(2.4)

diffusion:

$$v^* \frac{\partial C^*}{\partial y^*} = D^* \frac{\partial^2 u^*}{\partial y^*^2} - K^* C^*,$$

(2.5)

where $x^*$ and $y^*$ are the dimensional distances along and perpendicular to the plate, respectively. $u^*, v^*$ are the components of dimensional velocities along $x^*$ and $y^*$ directions, respectively, $\rho$ is the density, $V$ is the kinematic viscosity, $\nu$ is the kinematic rotational viscosity, $g$ is the acceleration of gravity, $\beta_f$ and $\beta_c$ are the coefficients of volumetric thermal and concentration expansion of the fluid, $\sigma_e$ is the fluid electrical conductivity, $B_0$ is the magnetic induction, $j^*$ is the micro inertia density, $\omega^*$ is the component of the angular velocity vector normal to the $x^* y^*$-plane, $\gamma$ is the spin gradient viscosity, $\alpha$ is the effective thermal diffusivity of the fluid, $k$ is the effective thermal conductivity, $C_p^*$ is the specific heat at pressure, $q^*$ is the radiative heat flux, $T^*$ is the dimensional temperature, $C^*$ is the dimensional concentration of the fluid, $K_j$ is the chemical reaction parameter and $D^*$ is the chemical molecular diffusivity. The second and third terms on the right hand side of the momentum equation (2.2) denotes thermal and concentration buoyancy effects and the forth is the MHD term. Also, the second term on the right hand side of the energy equation (2.4) represents the radiative heat flux.

Using the Rosseland approximation [21], the radiative heat flux in the $y^*$ direction is given by

$$q^* = -\frac{4\sigma \partial T'}{3\kappa \partial y^*},$$

(2.6)

where $\sigma$ and $k_\lambda$ are the Stefan-Boltzmann constant and the mean absorption coefficient, respectively.

We assume that the temperature differences within the flow are sufficiently small such that $T^4$ may be expressed as a linear function of the temperature. This is accomplished by expanding $T^4$ into the Taylor series about $T$ and neglecting higher terms, then

$$T^4 \approx 4T^3 - 3T^2$$

(2.7)

By substitution from equations (2.6) and (2.7) in equation (2.4), so

$$v^* \frac{\partial T^*}{\partial y^*} = k \frac{\partial^2 T^*}{\partial y^*^2} + 16\alpha \frac{\partial^2 T^*}{\partial y^*^2} + \nu \left( \frac{\partial u^*}{\partial y^*} \right)^2.$$  

(2.8)

The appropriate boundary conditions for the velocity, microrotation, temperature and concentration fields are

$$u^* = u_p^*, \quad \omega^* = -\frac{\partial u^*}{\partial y^*}, \quad T = T_w, \quad C^* = C_w^* \quad \text{at} \quad y^* = 0$$

$$u^* \rightarrow 0, \quad \omega^* \rightarrow 0, \quad T \rightarrow T_\infty, \quad C^* \rightarrow C_\infty^* \quad \text{as} \quad y^* \rightarrow \infty$$

(2.9)

The integration of the continuity equation (1) yields

$$\nu^* = -V_0,$$  

(2.10)

where $V_0$ is the scale of suction velocity which is a non-zero positive constant. The negative sign indicates that the suction is directed towards the plate.

It is convenient to employ the following non-dimensional variables:

$$y^* = \frac{V_0 y^*}{V}, \quad v^* = \frac{\nu}{V_0}, \quad \theta = \frac{T - T_w}{T_w - T_\infty}, \quad \mathcal{C} = \frac{C^* - C_w^*}{C_\infty^* - C_w^*},$$

$$\omega = \frac{\nu \omega^*}{V_0},$$

$$\Pr = \frac{k \alpha}{\nu} = \frac{c_p^*}{k}, \quad \mathcal{R} = \frac{kk_\lambda}{4\sigma T_w^3}, \quad \mathcal{B} = \frac{\beta}{\nu},$$

$$\mathcal{Q} = \frac{Q^* V_0^2}{c_p^*},$$

$$\mathcal{M} = \frac{\sigma B_0^2}{\rho V_0^2}, \quad \eta = \frac{1}{\gamma}, \quad \mathcal{G} = \frac{v \beta_f g (T_w - T_\infty)}{V_0^3},$$

$$\begin{align*}
\mathcal{C}_e &= \frac{\nu}{D^*}, \\
\Delta &= \frac{K^* v}{V_0^2}, \quad \mathcal{E}_i &= \frac{V_0^2}{c_p^* (T_w - T_\infty)}. \end{align*}$$

In view of Equations (2.10) and (2.11) the governing Equations (2.2), (2.3), (2.5) and (2.8) reduce to the following non-dimensional form:

$$\begin{align*}
(1 + \mathcal{B}) \frac{\partial^2 u^*}{\partial y^*^2} + \frac{\partial u^*}{\partial y^*} - Nu u^* &= \mathcal{G}_f \theta + \mathcal{G}_c + 2\beta \frac{\partial \omega^*}{\partial y^*}, \\
q^* &= \frac{4\sigma \partial T'}{3\kappa \partial y^*}.
\end{align*}$$

(2.12)
\[
\frac{\partial^2 \omega}{\partial y^2} + \frac{\eta \partial \omega}{\partial y} = 0, \quad (2.13)
\]
\[
\frac{\partial^2 \theta}{\partial y^2} + \Gamma \frac{\partial \theta}{\partial y} = -\Gamma Ec \left( \frac{\partial u}{\partial y} \right)^2, \quad (2.14)
\]
\[
\frac{\partial^2 C}{\partial y^2} + Sc \frac{\partial C}{\partial y} - Sc \Delta C = 0, \quad (2.15)
\]
where \( N = M + \frac{1}{K} \), \( \Gamma = \left( 1 - \frac{4}{3R + 4} \right) Pr \).

and \( Gr, Gc, M, K, Pr, Ec, Sc \) and \( \Delta \) are the thermal Grashof number, solutal Grashof number, magnetic field parameter, permeability parameter, Prandtl number, radiation parameter, Eckert number, Schmidt number and the chemical reaction parameter, respectively.

The boundary conditions (2.9) are then given by the following non-dimensional equations:

\[
u = U, \theta = 1, \omega = \frac{\partial u}{\partial y}, C = 1 \at \ y = 0
\]
\[
u \rightarrow 0, \theta \rightarrow 0, \omega \rightarrow 0, C \rightarrow 0 \as \ y \rightarrow \infty \quad (2.16)
\]

The mathematical statement of the problem is now complete and embodies the solution of Eqs. (2.12)-(2.15) and subject to boundary conditions (2.16).

**Solution of the problem**

The solution of Equation (2.15) subject to the boundary conditions (2.16).

\[
C(y) = e^{Ky}, \quad (2.17)
\]

where \( R_s = \frac{Sc}{2} \left[ 1 + \sqrt{1 + \frac{4\Delta}{Sc}} \right] \).

the problem posed in Eqs. (2.12)-(2.14) subjected to the boundary conditions presented in Equation (2.16) are highly non-linear, coupled equations and generally will involve a step by step numerical integration of the explicit finite difference scheme. However, analytical solutions are possible. Since viscous dissipation parameter is very small in most of the practical problems and therefore, we can advance an asymptotic expansion with as perturbation parameter for the velocity, microrotation and temperature as follows.

\[
u = \nu_0(y) + Ec \nu_1(y) + O(\varepsilon^2) + ...
\]
\[
\omega = \omega_0(y) + Ec \omega_1(y) + O(\varepsilon^2) + ...
\]
\[
\theta = \theta_0(y) + Ec \theta_1(y) + O(\varepsilon^2) + ...
\]

Substituting Eqs. (2.18) into Eqs. (2.12)-(2.14), equating the coefficients of the same power of \( \varepsilon \) and neglecting terms in \( \varepsilon^2 \) and higher order, we get

\[
(1 + \beta)u_0 + u_1 - Mu_0 = -G, \theta_0 - G, C - 2\beta \omega_0 \quad (2.19)
\]
\[
(1 + \beta)u_1 + u_1 - Mu_1 = -G, \theta_1 - 2\beta \omega_1 \quad (2.20)
\]
\[
\omega_0 + \eta \omega_0 = 0 \quad (2.21)
\]
\[
\omega_1 + \eta \omega_1 = 0 \quad (2.22)
\]
\[
\theta_0 + \Gamma \theta_0 = 0 \quad (2.23)
\]
\[
\theta_1 + \Gamma \theta_1 = -\Gamma \left( u_0 \frac{\partial u}{\partial y} \right) \quad (2.24)
\]

where a prime denotes differentiation with respect to \( y \).

With the corresponding boundary conditions

\[
u_0 = U, \nu_1 = 0, \omega_0 = -u_0, \omega_1 = -u_1, \theta_0 = 1, \theta_1 = 0, \at \ y = 0
\]
\[
u_0 = 0, \nu_1 = 0, \omega_0 \rightarrow 0, \omega_1 \rightarrow 0, \theta_0 \rightarrow 0, \theta_1 \rightarrow 0, \as \ y \rightarrow \infty \quad (2.25)
\]

Without going into details, the solutions of Eqs. (2.19)-(2.24) subject to Equation (2.25) can be shown to be

\[
u_0(y) = a_1 e^{-R_y} + a_2 e^{-R_y} + a_3 e^{-T_y} + a_4 e^{-\eta_y}
\]
\[
u_1(y) = \theta_0 e^{-R_y} + \theta_1 e^{-R_y} + \theta_2 e^{-R_y} + \theta_3 e^{-R_y} + \theta_4 e^{-R_y}
\]
\[
\omega_0(y) = C_1 e^{-\eta_y} + C_2 e^{-\eta_y}
\]
\[
\omega_1(y) = C_3 e^{-\eta_y} + C_4 e^{-\eta_y}
\]
\[
\theta_0(y) = e^{-T_y} + E \theta_1(y) + O(\varepsilon^2) + ...
\]
\[
C(y) = e^{Ky}
\]

where

\[
R_s = \frac{1}{2[1 + \beta]} \left[ 1 + \frac{4M}{1 + \beta} \right]
\]

and the remaining mathematical expressions involved in the above equations are given in appendix.

In view of the above solutions, the streamwise velocity, microrotation, temperature and concentration in the boundary layer become

\[
\nu_0(y) = a_1 e^{-R_y} + a_2 e^{-R_y} + a_3 e^{-R_y} + a_4 e^{-R_y}
\]
\[
\nu_1(y) = \theta_0 e^{-R_y} + \theta_1 e^{-R_y} + \theta_2 e^{-R_y} + \theta_3 e^{-R_y} + \theta_4 e^{-R_y}
\]
\[
\omega_0(y) = C_1 e^{-\eta_y} + E \theta_1(y) + O(\varepsilon^2) + ...
\]
\[
\omega_1(y) = C_3 e^{-\eta_y} + C_4 e^{-\eta_y}
\]
\[
\theta_0(y) = e^{-T_y} + E \theta_1(y) + O(\varepsilon^2) + ...
\]
\[
C(y) = e^{Ky}
\]

The skin-friction, the couple stress coefficient, the Nusselt number and the Sherwood number are important physical quantities for this type of boundary-layer flow. These parameters can be defined and determined as follows:

\[
C_f = \frac{\frac{\partial u}{\partial y}}{\frac{U}{\sqrt{f}}}, \quad = a_1 \frac{\partial \theta}{\partial y} + E \theta_1 \frac{\partial \theta}{\partial y} \quad (2.29)
\]

\[
= -E \left[ a_1 \frac{\partial \theta}{\partial y} + E \theta_1 \frac{\partial \theta}{\partial y} \right]
\]

\[
= -E \left[ a_1 \frac{\partial \theta}{\partial y} + E \theta_1 \frac{\partial \theta}{\partial y} \right]
\]

\[
= -\frac{\partial \theta}{\partial y} - E \theta_1 \frac{\partial \theta}{\partial y} \quad (2.30)
\]
Results and discussion

In order to study the behavior of velocity \( u \), microrotation \( \omega \), temperature \( \theta \) and concentration \( C \) fields, a comprehensive numerical computation is carried out for various values of the parameters that describe the flow characteristics viz., viscosity ratio \( \beta \), plate velocity \( U_p \), the thermal Grashof number \( Gr \), the solutal Grashof number \( Gc \), Prandtl number \( Pr \), the radiation parameter \( R \), Eckert number \( Ec \), Schmidt number \( Sc \) and chemical reaction parameter \( Kr \) and results reported in terms of graphs. The effect of dimensionless viscosity ratio \( \beta \), on the velocity and microrotation past a porous plate is presented in Fig. 1.

![Fig. 1 Velocity and Microrotation profiles for different values of \( \beta \)](image)

The numerical results show that the peak value of the velocity distribution across the boundary layer is smaller for a Newtonian fluid \( (\beta = 0) \) with the fixed flow and material parameters, as compared with a micropolar fluid. In addition, the value of the angular velocity on the porous plate increases as the viscosity ratio \( \beta \) increases.

Fig. 2 illustrates the variation of velocity and microrotation distribution across the boundary layer for the various values of the plate velocity \( U_p \). It is observed that the values of translational velocity and microrotation on the porous plate increase, as the plate moving velocity \( U_p \) increases.

![Fig. 2. Velocity and Microrotation profiles for different values of \( U_p \)](image)

The translational velocity and the microrotation profiles against spanwise coordinate \( y \) for different values of Grashof number \( Gr \), Solutal Grashof number \( Gc \) and Schmidt number \( Sc \) are described in Fig. 3. It is observed that the velocity increases as \( Gr \), \( Gc \) increase while it decreases as \( Sc \) increase, but opposite behavior is found to microrotation. Here the positive values of \( Gr \) corresponds to a cooling of the surface by natural convection.

![Fig. 3 Velocity and Microrotation profiles for different values of \( Gr \), \( Gc \) and \( Sc \)](image)

Fig. 4 shows the translational velocity and the microrotation profiles across the boundary layer for different values of \( Pr \) and \( R \). We observe that the effect of increasing values \( Pr \) or \( R \) in a decreasing the velocity and magnitude of microrotation.

![Fig. 4. Velocity and Microrotation profiles for different values of \( Pr \) and \( R \)](image)

For different values of the Schmidt number \( Sc \) and chemical reaction parameter \( Kr \) translational velocity and the microrotation profiles are plotted in Fig. 5. It is obvious that the effect of increasing values of \( Sc \) or \( Kr \) results in a decreasing velocity distribution across the boundary layer. Furthermore, the results show that the magnitude of microrotation on the porous plate is decreased as \( Sc \) or \( Kr \) increases.

![Fig. 5. Velocity and Microrotation profiles for different values of \( Sc \) and \( Kr \)](image)

The effects of the viscous dissipation parameter i.e., Eckert number on the velocity and microrotation are shown in Fig. 6. It is clear that from these results an increase in the values of Eckert number \( Ec \) leads to rise in the velocity and magnitude of microrotation.

![Fig. 6 Velocity and Microrotation profiles for different values of \( Ec \)](image)

Fig. 7 illustrates the influence of Eckert number \( Ec \) on the dimensionless temperature \( \theta \). It is observed that an increase in \( Ec \) leads to a fall in the velocity.
For different values of Prandtl number $Pr$ and radiation parameter $R$, the temperature profiles are plotted in Fig. 8. These results show that an increase in $Pr$ or $R$ results in a decrease thermal boundary layer thickness.

Fig. 9 shows typical variations in the concentration profiles for different values of the Schmidt number $Sc$ and the chemical reaction parameter $Kr$. It is clear from Fig. 9 that the concentration boundary layer thickness decreases as the Schmidt number $Sc$ and the chemical reaction parameter $Kr$.

References

Appendix

$$a_1 = U_p - \left( a_2 + a_3 + a_4 \right), \quad a_2 = \frac{-Gr}{(1+\beta)^2 \Gamma - \Gamma - M}$$

$$a_3 = \frac{\frac{-Gr}{(1+\beta)^2 \Gamma - \Gamma - M}}{\left(1+\beta\right)\eta C_1}$$

$$a_4 = \frac{-2\beta \eta C_1}{\left(1+\beta\right)\eta^2 - \eta - M}$$

$$a_5 = 1 - \sum_{n=0}^{15} a_n$$

$$C_1 = \frac{U \rho R_1 + a_3 \left(R_2 - R_1\right) + a_5 \left(\Gamma - R_1\right)}{1 + \theta \left(R_1 - \eta\right)}$$

$$a_6 = -\Gamma a_1^2 R_1, \quad a_7 = \Gamma a_2 R_2, \quad a_8 = -\Gamma a_3^2$$

$$a_9 = \frac{-2\Gamma a_3 R_2}{4R_2 - 2\Gamma}$$

$$a_{10} = \frac{-2\Gamma a_4}{(R_1 + R_2)^2 - \Gamma (R_1 + R_2)}$$

$$a_{11} = \frac{-2\Gamma a_1 a_3}{(R_1 + \Gamma)}$$

$$a_{12} = \frac{-2\Gamma a_4 a_5 R_2}{(R_1 + R_2)^2 - \Gamma (R_1 + R_2)}$$

$$a_{13} = \frac{-2\Gamma a_2 a_4}{(R_2 + \Gamma)^2}$$

$$a_{14} = \frac{-2\Gamma a_2 a_4 R_2}{(R_2 + \Gamma)^2}$$

$$a_{15} = \frac{-2\Gamma a_2 a_4}{(R_2 + \Gamma)}$$

$$b_1 = -\left(b_2 + b_3 + b_4 + b_5 + b_6 + b_7 + b_8 + b_9 + b_{10} + b_{11} + b_{12}\right)$$

$$b_2 = \frac{-Gra_1}{(1+\beta)^2 \Gamma^2 - \Gamma - M}$$

$$b_3 = \frac{-Gra_2}{4(1+\beta)^2 \Gamma^2 - 2\Gamma - M}$$

$$b_4 = \frac{-Gra_3}{4(1+\beta)^2 \Gamma^2 - 2\Gamma - M}$$

$$b_5 = \frac{-Gra_4}{4(1+\beta)^2 \Gamma^2 - 2\Gamma - M}$$
\[
\begin{align*}
    b_6 &= \frac{-\text{Gra}_9}{4(1 + \beta)\eta_1^2 - 2\eta - M}, \\
    b_7 &= \frac{-\text{Gra}_{10}}{(1 + \beta)(R_1 + R_2)^2 - (R_1 + R_2) - M}, \\
    b_8 &= \frac{-\text{Gra}_{11}}{(1 + \beta)(R_1 + \Gamma)^2 - (R_1 + \Gamma) - M}, \\
    b_9 &= \frac{-\text{Gra}_{12}}{(1 + \beta)(R_1 + \eta)^2 - (R_1 + \eta) - M}, \\
    b_{10} &= \frac{-\text{Gra}_{13}}{(1 + \beta)(R_2 + \Gamma)^2 - (R_2 + \Gamma) - M}, \\
    b_{11} &= \frac{-\text{Gra}_{14}}{(1 + \beta)(R_2 + \eta)^2 - (R_2 + \eta) - M},
\end{align*}
\]

\[
\begin{align*}
    b_{12} &= \frac{-\text{Gra}_{15}}{(1 + \beta)(\Gamma + \eta)^2 - (\Gamma + \eta) - M}, \\
    b_{13} &= \frac{2\beta C \eta}{(1 + \beta)\eta^2 - \eta - M} = \theta_2 C_2, \\
    C_3 &= b_2 [\Gamma - R] + b_3 |2R_2 - R| + b_5 |R - R| + b_6 |2\eta - R| + b_7 R_2 \\
&+ b_8 \Gamma + b_9 \eta + b_{10} |R_1 - R + \Gamma| + b_{11} |R_2 - R + \eta| + b_{12} |\Gamma - R + \eta|, \\
    C_2 &= \frac{C_3}{(1 + \theta_2 (R_1 - \eta))}.
\end{align*}
\]