Two summation formulae based on half argument involving contiguous relation
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ABSTRACT
The main objective of this paper is to establish two summation formulae based on half argument involving Contiguous Relation. The results derived in this paper are of general character.
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Introduction
The Pochhammer’s symbol
\[(a, k) = (a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} \quad (1)\]

Generalized Gaussian Hypergeometric function of one variable
\[\sum_{k=0}^{\infty} \frac{(a_1)_k(a_2)_k...a_n(b_1)_k(b_2)_k...b_n(z)}{(b_1)_k(b_2)_k...b_n(z)} \quad (2)\]

where the parameters \(b_1, b_2, ... b_n\) are neither zero nor negative integers and \(a, b\) are non-negative integers.

Contiguous Relations
[Andrews p.363(9.16), E.D. p.51(10), H.T.F.L. p.103(32)]
\[(a-b)F_1(a; b; c; z) = aF_1(a+1; b; c; z) - bF_1(a, b+1; c; z) \quad (3)\]
\[[Abramowitz p.558(15.2.19)]
\|(a-b) (1-z)F_1(a, b; c; z) = (c-b)F_1(a, b-1; c; z) + (a-c)F_1(a-1, b; c; z) \quad (4)\]

Recurrence relation
\[\Gamma(z+1) = z \Gamma(z) \quad (5)\]

A New Summation Formula [2]
\[2F_1(a, b; \frac{a+b+1}{2}; z) = 2^{b-1} \frac{(\frac{a+b+1}{2}) \Gamma(\frac{a+1}{2})}{\Gamma(\frac{b+1}{2})} \quad (6)\]

Main Summation Formulae
For both the results \(a \neq b\)
For \(a<1\) and \(a>12\)
\[2F_1(a, b; \frac{a+b+1}{2}; z) = 2^{b-1} \frac{(\frac{a+b+1}{2}) \Gamma(\frac{a+1}{2})}{\Gamma(\frac{b+1}{2})} \quad (7)\]
C derivations of summation formulae (7) to (8):

Derivation of (7): Substituting \( c = \frac{a-b-1}{2} \) and \( z = \frac{1}{2} \) in equation (4), we get

\[
\begin{align*}
( a-b ) \ \frac{\Gamma ( a-b ) \Gamma ( a ) \Gamma ( a-b-1 )}{\Gamma ( a ) \Gamma ( a-b )} = & \quad ( a-b-1 ) \ \frac{\Gamma ( a-b-1 ) \Gamma ( a ) \Gamma ( a-b-1 )}{\Gamma ( a ) \Gamma ( a-b-1 )} \\
+ ( a+b+1 ) \ \frac{\Gamma ( a+b+1 ) \Gamma ( a-b-1 )}{\Gamma ( a-b-1 ) \Gamma ( a+b+1 )} \\
\end{align*}
\]

Now with the help of the derived result from equation (6), we get

\[
\begin{align*}
\text{L.H.S} = & \quad ( a-b-1 ) \ 2^{b-2} \ \frac{\Gamma ( a-b ) \Gamma ( a ) \Gamma ( a-b-1 )}{\Gamma ( a ) \Gamma ( a-b )} \\
+ & \quad \frac{\Gamma ( a-b ) \Gamma ( a ) \Gamma ( a-b-1 )}{\Gamma ( a ) \Gamma ( a-b )} \\
+ & \quad \frac{\Gamma ( a-b ) \Gamma ( a ) \Gamma ( a-b-1 )}{\Gamma ( a ) \Gamma ( a-b )} \\
+ & \quad \frac{\Gamma ( a-b ) \Gamma ( a ) \Gamma ( a-b-1 )}{\Gamma ( a ) \Gamma ( a-b )} \\
+ & \quad \frac{\Gamma ( a-b ) \Gamma ( a ) \Gamma ( a-b-1 )}{\Gamma ( a ) \Gamma ( a-b )} \\
\end{align*}
\]

On simplification, we get the result (7)

Similarly, we can prove the other result (8).

References

7. Rainville, E. D.; The contiguous function relations for $\text{pF}_{\text{q}}$ with applications to Bateman’s $J_{\text{a},\text{b}}$ and Rice’s $H_{\text{a}} (\zeta, \text{p}, \text{v})$, *Bull. Amer. Math. Soc.*, 51(1945), 714-723.